On Bi-Modal Constrained Coding

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Abstract—Bi-modal (respectively, multi-modal) constrained coding refers to an encoding model whereby a user input block can be mapped to two (respectively, multiple) codewords. In current storage applications, such as optical disks, multi-modal coding allows to achieve DC control, in addition to satisfying the runlength limited (RLL) constraint specified by the recording channel. In this work, a study is initiated on bi-modal fixed-length constrained encoders. Necessary and sufficient conditions are presented for the existence of such encoders for a given constraint. It is also shown that under somewhat stronger conditions, one can guarantee a bi-modal encoder with finite decoding delay.

Index Terms—Approximate eigenvectors, Bi-modal encoders, Constrained codes, Multi-modal encoders, Parity-preserving encoders.

I. INTRODUCTION

Runlength limited (RLL) coding is widely employed in magnetic and optical storage in order to mitigate the effects of inter-symbol interference and clock drifting [9]. The encoder typically takes the form of a finite-state machine, which maps a sequence of input p-bit blocks into a sequence of output q-bit codewords, so that the concatenation of the generated codewords satisfies the RLL constraint. In most applications, the coding scheme also provides DC control (or, more generally, suppression of the low frequencies). This is achieved by bi-modal (respectively, multi-modal) encoding, allowing some (or all) input p-blocks to be mapped by the encoder to two (respectively, multiple) codewords, and then, during the encoding process, selecting the codeword that yields the best DC suppression. In one implementation of this strategy, the input sequence is sub-divided into non-overlapping windows (each consisting of one or more p-blocks), and each window can be mapped by the encoder to two output sequences over \{0, 1\} that have different parities (i.e., modulo-2 sums). The generated output binary sequence is then transformed (“pre-coded”) into sequences over the bipolar alphabet \{+1, -1\}, with the binary 1s corresponding to positions of the bipolar sign changes [9, p. 52]. DC control is achieved by selecting the output sequence that minimizes the DC contents [14, p. 29]. A parity-preserving encoder is one embodiment of this approach and is part of the Blu-ray standard. In such an encoder, the parity of the input sub-sequence within each window is preserved at the generated output sequence; one bit in each window is then reserved to set the parity of (the input window and) the output sequence, thereby controlling the DC contents [9, §11.4.3], [11], [15], [16], [17], [19]. Another implementation of DC control via multi-modal encoding maps the input sequence (without precoding) into bipolar sequences with different disparities (i.e., different signs of their sums of entries). Guided scrambling, which is a known paradigm for DC control enhancement (see, for example, [10]), can be combined with multi-modal encoding [9, Ch. 10].

Most constructions so far of multi-modal encoders and, in particular, of parity-preserving encoders, were obtained by ad-hoc methods. The purpose of this work is to initiate a study of bi-modal (and multi-modal) encoders, starting with the special case of fixed-length finite-state encoders. We provide a formal definition of our setting in Section I-B below, following a summary (in Section I-A) of some background and definitions which are taken from [14]. The main result of the paper, which we state in Section I-C, is a necessary and sufficient condition for the existence of a bi-modal encoder. We prove the necessity part in Section II, followed by a construction method of bi-modal encoders in Section III, thereby establishing sufficiency. The more general variable-length model is a subject of future work [18] and is briefly discussed in Section IV.

A. Background

A (finite labeled directed) graph is a graph \( G = (V,E,L) \) with a nonempty finite state (vertex) set \( V = V(G) \), finite edge set \( E = E(G) \), and edge labeling \( L : E \to \Sigma \). The constraint presented by \( G \), denoted \( S(G) \), is the set of words over \( \Sigma \) that are generated by finite paths in \( G \). The set of words that are generated by finite paths that start at a given state \( u \in V(G) \) is called the follower set of \( u \) and is denoted \( F_G(u) \).

A graph \( G \) is deterministic if all outgoing edges from a state are distinctly labeled. Every constraint has a deterministic presentation. A graph \( G \) is lossless if no two paths with the same initial state and the same terminal state generate the same word. The anticipation \( A(G) \) of \( G \) is the smallest nonnegative integer \( a \) (if any) such that all paths that generate any given word of length \( a+1 \) from any given state in \( G \) share the same first edge (thus, the anticipation of deterministic graphs is 0; generally, having finite anticipation is a stronger property than losslessness). A graph is \((m,a)\)-definite if all paths that generate a given word of length \( m+a+1 \) share the same \((m+1)\)st edge. A graph is said to have finite memory \( \mu \) if \( \mu \) is the smallest nonnegative integer (if any) such that all paths of length \( \mu \) that generate the same word terminate in the same state.
A graph \( G \) is irreducible if it is strongly-connected. Every graph decomposes uniquely into irreducible components, and at least one such component must be an irreducible sink (i.e., with no outgoing edges to another component). The period of an irreducible graph (with at least one edge) is the greatest common divisor of the lengths of its cycles, and such a graph is primitive if its period is 1. A constraint \( S \) is irreducible if it can be presented by a deterministic irreducible graph (in which case the smallest such presentation is unique).

The power \( G^t \) of a graph \( G \) is the graph with the same set of states \( V(G) \) and edges that are the paths of length \( t \) in \( G \); the label of an edge in \( G^t \) is the length-\( t \) word generated by the path. For \( S = S(G) \) the power \( S^t \) is defined as \( S(G^t) \).

Given a constraint \( S \) over an alphabet \( \Sigma \) and a lossless presentation \( G \) of \( S \), the capacity of \( S \) is defined by \( \operatorname{cap}(S) = \lim_{t \to \infty} \frac{1}{t} \log_2 |S \cap \Sigma^t| \). The limit indeed exists, and it is known that \( \operatorname{cap}(S) = \log_2 \lambda(A_G) \) where \( \lambda(A_G) \) denotes the spectral radius (Perron eigenvalue) of the adjacency matrix \( A_G \).

Given a constraint \( S \) and a nonnegative\(^1 \) integer \( n \), an \((S,n)\)-encoder is a lossless graph \( E \) such that \( S(E) \subseteq S \) and each state has out-degree \( n \). An \((S,n)\)-encoder exists if and only if \( \log_2 n \leq \operatorname{cap}(S) \). In a tagged \((S,n)\)-encoder, each edge is assigned an input tag from a finite alphabet of size \( n \), such that edges outgoing from the same state have distinct tags. The anticipation (if finite) of an encoder determines its decoding delay: given the current encoder state, the anticipation specifies how far one needs to look-ahead at a sequence generated from that state in order to recover the first edge (and, thus, the tag of that edge) in the encoder path that generated that sequence. A tagged encoder is \((m,a)\)-sliding-block decodable if all paths that generate a given word of length \( m+a+1 \) share the same tag on their \((m+1)\)st edges (thus, an \((m,a)\)-definite encoder is \((m,a)\)-sliding-block decodable for any tagging of its edges).

A (tagged) rate \( p \) : \( q \) encoder for a constraint \( S \) is a (tagged) \((S^q,2^p)\)-encoder (the tags are then assumed to be from \{0,1\}\(^q\)). A rate \( p \) : \( q \) parity-preserving encoder for a constraint \( S \) over the binary alphabet is an \((S^q,2^{p-1},2^{q-1})\)-encoder, where we partition the alphabet \( S \cap \{0,1\}^q \) into even and odd symbols according to their parity. In particular, a rate \( p : q \) parity-preserving encoder for \( S \) is a (tagged) \((S^q,2^{p-1},2^{q-1})\)-encoder and, conversely, any \((S^q,2^{p-1},2^{q-1})\)-encoder can be tagged so that it is parity preserving. In the disparity-based DC control scheme, the constraint \( S \) is over the bipolar alphabet and we partition the set of length-\( q \) words of \( S \) according to the sign of their disparity (zero-disparity words can be treated arbitrarily, but see Footnote 2). In general, by a rate \( p : q \) bi-modal encoder for a constraint \( S \) we hereafter mean an \((S^q,2^{p-1},2^{q-1})\)-encoder with respect to a prescribed partition of the set of length-\( q \) words of \( S \).

**Example 1.** Let \( S \) be the constraint over \( \Sigma = \{a,b,c,d\} \) which is presented by the graph \( E \) in Figure 1. The graph \( E \) is actually a deterministic \((S,4)\)-encoder. Moreover, it is an \((S,2,2)\) encoder if we assume the partition \( \Sigma_0 = \{a,b\} \) and \( \Sigma_1 = \{c,d\} \); the \((S,2)\)-encoders \( E_0 \) and \( E_1 \) are shown in Figures 2 and 3, respectively.

When studying \((S(G),n_0,n_1)\)-encoders, there is no loss of generality in assuming that both \( G \) and the encoder are irreducible. Indeed, if \( E \) is an \((S(G),n_0,n_1)\)-encoder, then an irreducible sink of \( E \) is an \((S',n_0,n_1)\)-encoder, where \( S' \) is an irreducible constraint presented by some irreducible component of \( G \) (see [14, Lemma 2.9]). Note that \( G_0, G_1, E_0, \) and \( E_1 \) may still be reducible even when \( G \) and \( E \) are irreducible.

\(^2\)Moreover, we will also consider in Section III-D the extension where \( \Sigma_0 \) and \( \Sigma_1 \) are not necessarily disjoint (yet still cover all the elements of \( \Sigma \)), as this extension fits better the disparity-based DC control scheme; in this case, \( \Sigma_0 \) and \( \Sigma_1 \) represent sets of bipolar \( q \)-blocks of non-negative and non-positive disparities, respectively and, so, they may intersect on \( q \)-blocks with zero disparity.
is also a rational solution, and, therefore, there is a nonzero are integers, if there is a real feasible solution (for Example 1, the graph \( G \) is irreducible (e.g., in Example 1, the graph \( E \) is irreducible, while \( E_0 \) is not).

C. Statement of main result

Next is a statement of our main result.

**Theorem 1.** Let \( S \) be an irreducible constraint, presented by an irreducible deterministic graph \( G \), and let \( n_0 \) and \( n_1 \) be positive integers. Then there exists an \((S,n_0,n_1)\)-encoder if and only if \( \mathcal{X}(A_{G_0},n_0) \cap \mathcal{X}(A_{G_1},n_1) \neq \emptyset \).

We prove Theorem 1 in Sections II (necessity) and III (sufficiency). Hereafter, we use the notation \( \mathcal{X}(A_{G_0},n_0) \cap \mathcal{X}(A_{G_1},n_1) \) for the intersection \( \mathcal{X}(A_{G_0},n_0) \cap \mathcal{X}(A_{G_1},n_1) \).

In view of Theorem 1, finding the possible pairs \((n_0,n_1)\) for which an \((S(G),n_0,n_1)\)-encoder exists for a given \( G \) and partition \( \{\Sigma_0,\Sigma_1\} \) requires a method for deciding whether \( \mathcal{X}(A_{G_0},n_0) \) and \( \mathcal{X}(A_{G_1},n_1) \) share common vectors. This decision problem can be recast as a linear programming problem, namely, deciding whether there is a real vector \( x \) that satisfies the following constraints:

\[
\begin{align*}
(A_{G_0} - n_0 I) x & \geq 0 \\
(A_{G_1} - n_1 I) x & \geq 0 \\
1^\top \cdot x & = 1 ,
\end{align*}
\]  

where \( 0 \) and \( 1 \) stand for the all-zero and all-1 column vectors and \((\cdot)^\top\) denotes transposition. Since all the coefficients in (1) are integers, if there is a real feasible solution \( x \) then there is also a rational solution, and, therefore, there is a nonzero integer solution that satisfies the (first) three inequalities in (1).

There are known polynomial-time algorithms for solving linear programming problems, such as Karmarkar’s algorithm [12], but it would be interesting to find a more direct method, tailored specifically to the constraints (1), for determining whether \( \mathcal{X}(A_{G_0},A_{G_1},n_0,n_1) \) is nonempty (see also the modified Franaszek algorithm in Figure 6 in the sequel). In comparison, recall that in the context of ordinary \((S,n)\)-encoders, the question of interest is whether \( \mathcal{X}(A_G,n) \) is nonempty, which, in turn, is equivalent to asking whether \( n \leq \lambda(A_G) \).

D. Going to powers of the constraint

Next, we discuss the effect of going to powers of a constraint, namely, attempting to construct \((S^t,n_0,n_1)\)-encoders, for increasing values of \( t \). To this end, we first need to define the even and odd symbols in \( \Sigma^t \), which is the alphabet\(^3\) of \( S^t \), given a partition \( \{\Sigma_0,\Sigma_1\} \) of \( \Sigma \). We adopt a definition that fits the parity-based scheme for DC control: we say that the parity-based scheme for DC control: we say that the parity of the symbols in \( \Sigma^t \) is even (respectively, odd) if it contains an even (respectively, odd) number of symbols from \( \Sigma_1 \) (i.e., the parity of \( w \) is the modulo-2 sum of the parities of the symbols in \( w \)). The set of even (respectively, odd) words in \( \Sigma^t \) will be denoted by \( \Sigma^t \) (respectively, \( \Sigma^t \)).

It turns out that in most cases, we can approach the capacity of \( S \) with bi-modal encoders if we let \( t \) increase. Note, however, that such an increase may sometimes be necessary, even when the capacity of \( S \) is \( \log_2(n_0+n_1) \) (see Example 2 below). This presents a distinction between bi-modal encoders and ordinary ones: when \( \text{cap}(S) = \log_2 n \), capacity is always attained by ordinary encoders already for \( t = 1 \).

Specifically, we have the following result.

**Theorem 2.** Let \( G \) be a deterministic primitive graph, having at least one edge with an even label and one edge with an odd label. Then there exists an infinite sequence of nonnegative integers \( n^{(1)},n^{(2)},\ldots \) such that \((S(G^t),n^{(t)},n^{(t)})\)-encoders exist and

\[
\lim_{t \to \infty} (\log_2 n^{(t)})/t = \text{cap}(S(G)).
\]

**Proof.** By assumption, there are edges \( e_0 = v_0 \rightarrow v_0' \) and \( e_1 = v_1 \rightarrow v_1' \) in \( G \) such that \( a \) is even and \( b \) is odd \((v_0 \text{ and } v_1 \text{ may be the same state, and so may } v_0' \text{ and } v_1')\). Also, since \( G \) is primitive, there exists a real \( \kappa > 0 \) such that \( \lambda_{G}^{(t)} > \kappa \cdot \lambda^t \) for any sufficiently large \( t \) and for every \( u, v \in V(G) \), where \( \lambda = \lambda(A_G) \) (see [14, §3.3.4]).

Fix \( u^* \) to be some \( G \)-state. We show that there are two paths in \( G \) of the same length \( \ell^* \) that start at \( u^* \) and have different parities. Let \( \ell \) be sufficiently large such that \( \lambda_{G}^{(t)} > 0 \) (componentwise), and consider all paths of length \( \ell \) starting at \( u^* \). If two of them have different parities, we are done, with \( \ell^* \) taken as \( \ell \). Otherwise, for \( b \in [2], \) let \( \pi_b \) be a path of length \( \ell \) in \( G \) that starts at \( u^* \) and terminates in \( v_b \). The paths \( \pi_0, \pi_1 \), of length \( \ell^* = \ell+1 \), then have different parities.

Now, if \( t - \ell^* \) is sufficiently large, there are at least \( n^{(t)} = [\kappa \cdot \lambda^t - \ell]^{t-\ell^*} \) paths of length \( t - \ell^* \) from any state \( u \in V(G) \) to \( u^* \). It follows that from any state \( u \in V(G) \), there are at least \( n^{(t)} \) paths of length \( t \) that generate even words,

\(^3\)The effective alphabet of \( S^t \) is the subset \( S \cap \Sigma^t \) of \( \Sigma^t \).
and at least \( n^{(t)} \) paths of that length that generate odd words. This means that \( G^t \) contains a subgraph which is an \((S(G^t), n^{(t)}, n^{(t)})\)-encoder. The result follows by noticing that 
\[
\lim_{t \to \infty} \frac{\log_2(n^{(t)})}{t} = \log_2 \lambda = \text{cap}(S(G)).
\]

**Remark 1.** The requirement in Theorem 2 of having both even and odd labels in \( G \) is necessary. Indeed, if all the labels in \( G \) were even, then so would have to be all the labels in the encoder, implying that \( n^{(t)} = 0 \). Similarly, if all the labels in \( G \) were odd, then all the labels in the encoder would have to be even (respectively, odd) for even (respectively, odd) \( t \), again implying that \( n^{(t)} = 0 \).

For a deterministic graph \( G \) and a positive integer \( t \), we denote by \( n_{\text{max}}(G, t) \) the largest integer \( n \) for which there exist \((S(G^t), n, n)\)-encoders, and define the largest possible coding ratio attainable by such encoders by
\[
\rho(G, t) = \frac{\log_2(2 n_{\text{max}}(G, t))}{t}.
\]

**Example 2.** Let \( G \) be the graph in Figure 4, where
\[
\Sigma_0 = \{a, b\} \quad \text{and} \quad \Sigma_1 = \{c, d\}.
\]

We have \( \lambda(A_G) = 2 \), and the matrices \( A_{G_0} \) and \( A_{G_1} \) are

\[
A_{G_0} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{G_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Letting \( S = S(G) \), we find conditions under which there exist \((S^t, n, n)\)-encoders, for various values of \( t \). Starting with \( t = 1 \), in this case, \( \mathcal{X}(A_{G_0}, A_{G_1}, n, n) = \emptyset \) for every \( n > 0 \). Indeed, when \( n > 0 \), any \( x = (x_0, x_1)^T \in \mathcal{X}(A_{G_0}, n) \) must have \( x_1 = 0 \) (since \( \beta \) has no outgoing edges in \( G_0 \)), but then \( A_{G_1} x \geq n x \) implies that \( x_0 = 0 \). Hence, \( n_{\text{max}}(G, 1) = 0 \) and \( \rho(G, 1) = -\infty \).

Next, we turn to the second power of \( G \). Here \( S \cap (\Sigma^2)_0 = \{aa, ab, cd, dc\} \), \( S \cap (\Sigma^2)_1 = \{ac, bd, da, db\} \), and, respectively,
\[
A_{G^2}_0 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{G^2}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

We must have \( n \leq \lambda(A_{G^2_1}) = 2 \). Checking first the case \( n = 2 \), any \( x \in \mathcal{X}(A_{G^2_1}, 2) \) must be a multiple of \((1 1)^T\), which is a (true) eigenvector of \( A_{G^2_1} \) associated with the eigenvalue \( 2 \) [14, Theorem 5.4]. Yet \((1 1)^T \notin \mathcal{X}(A_{G^2_0}, 2) \), so we must have \( n \leq 1 \), and it is easily seen that \((1 1)^T \in \mathcal{X}(A_{G^2_0}, A_{G^2_1}, 1, 1) \neq \emptyset \). Therefore, by Theorem 1 there exists an \((S^2, 1, 1)\)-encoder (i.e., a rate 1 : 2 bi-modal encoder for \( S \) with respect to the partition \((\Sigma^2)_0, (\Sigma^2)_1 \)). In this case, \( n_{\text{max}}(G, 2) = 1 \) and \( \rho(G, 2) = 1/2 \).

Turning to the third power of \( G \), here
\[
S \cap (\Sigma^3)_0 = \{aab, cda, cdb, dcd, acd, bdc, dac, dbd\} \quad \text{and} \quad S \cap (\Sigma^3)_1 = \{ace, abd, cde, dcd, bda, bbd, dba, dab\},
\]
and, respectively,
\[
A_{G^3} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_{G^3_1} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.
\]

We must have \( n \leq \lambda(A_{G^3_1}) = 4 \), and we again rule out the case \( n = 4 \) by observing that \( \mathcal{X}(A_{G^3_1}, 4) \) consists of multiples of \((3 1)^T \), yet this vector is not in \( \mathcal{X}(A_{G^3_0}, 4) \). Hence, we must have \( n \leq 3 \), and since \((3 1)^T \in \mathcal{X}(A_{G^3_0}, A_{G^3_1}, 3, 3) \) we get by Theorem 1 that there exists an \((S^3, 3, 3)\)-encoder. Thus, \( n_{\text{max}}(G, 3) = 3 \) and \( \rho(G, 3) = (\log_2 6)/3 \).

In general, using the equalities
\[
A_{G^t} = A_{G_0} A_{G^t-1} + A_{G_1} A_{G^t-1} \quad \text{and} \quad A_{G^t_1} = A_{G_0} A_{G^t-1} + A_{G_1} A_{G^t-1} \quad \text{it can be shown by induction on} \ t
\]
that
\[
A_{G^t} = \frac{1}{6} \begin{pmatrix} 2^{t+1} + 3 + (-1)^t & 2^{t+1} - 2(-1)^t \\ 2^t - 3 + (-1)^t & 2^t + 2(-1)^t \end{pmatrix}
\]
and
\[
A_{G^t_1} = \frac{1}{6} \begin{pmatrix} 2^{t+1} - 3 + (-1)^t & 2^{t+1} - 2(-1)^t \\ 2^t + 3 + (-1)^t & 2^t + 2(-1)^t \end{pmatrix},
\]
with \( \lambda(A_{G^t}) = \lambda(A_{G^t_1}) = 2^{t-1} \) and respective eigenvectors
\[
x_0 = \begin{pmatrix} 2^{t+1} - 2(-1)^t \\ 2^t - 3 + (-1)^t \end{pmatrix} \quad \text{and} \quad x_1 = \begin{pmatrix} 2^{t+1} - 2(-1)^t \\ 2^t + 3 + (-1)^t \end{pmatrix}.
\]
Clearly, each of the sets \( \mathcal{X}(A_{G^t_0}, n) \) and \( \mathcal{X}(A_{G^t_1}, n) \) is empty if and only if \( n > 2^{t-1} \). Their intersection, however, is empty also when \( n = 2^{t-1} \): for \( b \in [2] \), the elements of \( \mathcal{X}(A_{G^t_0}, 2^{t-1}) \) are (true) eigenvectors of \( A_{G^t_0} \) associated with the Perron eigenvalue \( 2^{t-1} \), namely, they are scalar multiples of \( x_b \), yet \( x_0 \) and \( x_1 \) are linearly independent. We therefore conclude that \( \mathcal{X}(A_{G^t_0}, A_{G^t_1}, n_0, n_1) \neq \emptyset \) only when \( \max\{n_0, n_1\} \leq 2^{t-1} \) and \( n_0 + n_1 < 2^t \); in particular, there can be no \((S^t, n_0, n_1)\)-encoder when \( \log_2(n_0 + n_1) = t \) (and this holds also when \( n_0 \neq n_1 \)). It follows that there can be no rate \( t \) : \( t \) bi-modal encoder for \( S \) (with respect to the partition \((\Sigma^t)_0, (\Sigma^t)_1 \)) for any positive integer \( t \).

On the other hand, for \( n^{(t)} = 2^{t-1} - 1 \), we do have \((2 1)^T \in \mathcal{X}(A_{G^t_0}, A_{G^t_1}, n^{(t)}, n^{(t)}) \). Therefore, \( \rho(G, t) = (1/t) \log_2(2^{t-2} - 1) \), and \( \lim_{t \to \infty} \rho(G, t) = 1 = \log_2 \lambda(A_G) \), so the rate can approach the capacity value of \( 1 \) as \( t \) increases (yet can never attain it).

**Example 3.** Let \( G \) be as in Example 2, except that now
\[
\Sigma_0 = \{a\} \quad \text{and} \quad \Sigma_1 = \{b, c, d\}.
\]

It can be shown by induction on \( t \) that in this case,
\[
A_{G^t} = \frac{1}{3} \begin{pmatrix} 2^{t+1} + (-1)^t & 0 \\ 0 & 2^t + 2(-1)^t \end{pmatrix}
\]
and
\[
A_{(G^t)_1} = \frac{1}{3} \begin{pmatrix} 2^t & 0 \\ 2^t - (1)^t & 2^{t+1} - 2(-1)^t \end{pmatrix}.
\]
Letting
\[
n^{(t)} = (A_{(G^t)_0})_{\beta, \beta} = (1/3) (2^t + 2(-1)^t)
\]
and
\[
\lambda^{(t)} = \lambda(A_{(G^t)_1}) = (\sqrt{2}/3) (2^t - (-1)^t),
\]
it can be verified that \(\mathcal{X}(A_{(G^t)_0}, A_{(G^t)_1}, n_0, n_1) \neq \emptyset\) only when \(n_0 \leq n^{(t)}\) and \(n_1 \leq \lambda^{(t)}\). In particular, since \(n^{(t)} + \lambda^{(t)} < 2t\), there can be no \((S^t, n_0, n_1)\)-encoder when \(\log_2(n_0 + n_1) = t\).

On the other hand, when \(t > 2\), we do have \((3, 2)^T \in \mathcal{X}(A_{(G^t)_0}, A_{(G^t)_1}, n^{(t)}, n^{(t)})\). Hence, \(n_{\max}(G, t) = n^{(t)}\) and, similarly to Example 2, \(\rho(G, t) = (1/t) \log_2(2n^{(t)}) \leq 1\); we can approach the capacity value of (1), although we will never attain it.

Note that while the graphs \(G^t\) are primitive for all positive integers \(t\), the graphs \((G^t)_0\) in this example are reducible and \((G^t)_1\) are irreducible with period 2.

The result of Theorem 2 can be generalized to the non-primitive case as follows. Let \(G^p\) be an irreducible deterministic graph with period \(p\). Then \(G^p\) decomposes into \(p\) irreducible components \(G_0, G_1, \ldots, G_{p-1}\), with each \(G_i\) being the induced subgraph on a congruence class \(C_i\) in \(V(G)\). Suppose that for some \(i\), there are two paths of length \(p\) that start at states in \(C_i\) and generate words with different parities. Under these conditions, the subgraph \(G_i\) will satisfy the conditions of Theorem 2.

Theorem 2 (along with the previous examples) focused on \((S, n_0, n_1)\)-encoders where \(n_0 = n_1\). While this case suits the motivation of rate \(p:q\) bi-modal encoders (where \(n_0 = n_1 = 2^{p-1}\)), there seems to be a merit in studying the more general case as well (see Example 5 in Section III-B).

Example 4. Let \(G\) be a deterministic graph with the adjacency matrix
\[
A_G = \begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix},
\]
and let \(\Sigma_0\) and \(\Sigma_1\) be such that
\[
A_{G_0} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad A_{G_1} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}.
\]
Then
\[
A_{(G^2)_0} = \begin{pmatrix} 12 & 9 \\ 23 & 32 \end{pmatrix} \quad \text{and} \quad A_{(G^2)_1} = \begin{pmatrix} 12 & 15 \\ 17 & 8 \end{pmatrix}.
\]
It can be verified that \(\lambda(A_{G^2}) = 64\), \(\lambda(A_{(G^2)_0}) \approx 39.5\), and \(\lambda(A_{(G^2)_1}) \approx 26.1\). Figure 5 shows the boundary of the region of all pairs \((n_0, n_1)\) for which \(\mathcal{X}(A_{(G^2)_0}, A_{(G^2)_1}, n_0, n_1) \neq \emptyset\).

II. Necessary Condition

In this section we prove the “only if” part in Theorem 1, recalled next.

Theorem 3. Let \(S\) be an irreducible constraint, presented by an irreducible deterministic graph \(G\), and let \(n_0\) and \(n_1\) be positive integers. Then, an \((S, n_0, n_1)\)-encoder exists, only if \(\mathcal{X}(A_{G_0}, A_{G_1}, n_0, n_1) \neq \emptyset\), namely only if there exists a nonnegative nonzero integer vector \(x\) such that
\[
A_{G_0} x \geq n_0 x \quad \text{and} \quad A_{G_1} x \geq n_1 x.
\]

The proof of the theorem is a refinement of the proof of Theorem 3 in [13] (or Theorem 7.2 in [14]), where it was shown, inter alia, that the existence of an \((S, n)\)-encoder implies the existence of an \((A_G, n)\)-approximate eigenvector. Since the existence of an \((S, n_0, n_1)\)-encoder implies the existence of \((S(G^0), n_0)\)-encoders for \(b \in [2]\), it also implies that \(\mathcal{X}(A_{G_0}, n_0)\) and \(\mathcal{X}(A_{G_1}, n_1)\) are both nonempty sets (and, so, \(n_0 \leq \lambda(A_{G_0})\) and \(n_1 \leq \lambda(A_{G_1})\)). Theorem 3 states that their intersection must be nonempty too.

Proof of Theorem 3. Suppose that an \((S, n_0, n_1)\)-encoder exists, and fix \(E\) to be such an encoder. Next, we follow the three steps of the proof in [13]: step (a) is identical to the respective step in that proof, and so are the definitions of \(H', c\), and \(x\) in the remaining steps.

(a) Construct a deterministic graph \(H\) which presents \(S(E)\). For any word \(w\) and state \(v \in V(E)\), let \(T_E(w, v)\) denote the subset of states in \(E\) which are accessible from \(v\) by paths in \(E\) which generate \(w\) (when \(w\) is the empty word define \(T_E(w, v) = \{v\}\)). The states of \(H\) are defined as the distinct nonempty subsets \(\{T_E(w, v)\}_{w,v}\) of \(V(E)\), and we endow \(H\) with an edge \(Z \rightarrow Z'\) if and only if there exists a state \(v \in V(E)\) and a word \(w\) such that \(Z = T_E(w, v)\) and \(Z' = T_E(wv, v)\). The graph \(H\) is deterministic by construction, and by [14, Lemma 2.1] we have \(S(H) = S(E)\). For the rest of the proof, fix \(H'\) to be any irreducible sink of \(H\).

(b) Define a positive integer vector \(c\) such that \(A_{H'} c = n_0 c\) and \(A_{H'} c = n_1 c\). Recalling that each state \(Z \in V(H')\) is a subset of \(V(E)\), let \(c_Z = |Z|\) denote the number of \(E\)-
states in $Z$ and let $c$ be the positive integer vector defined by $c = (c_Z)_{Z \in V(H')}$. We claim that

$$A_{H'_u}c = n_0c \quad \text{and} \quad A_{H'_1}c = n_1c.$$  \hspace{1cm} (6)

Fix a parity $b \in \{2\}$, and consider a state $Z \in V(H')$. Since $E_b$ has out-degree $n_b$, the number of edges in $E_b$ outgoing from the subset of states $Z \subseteq V(E)$ is $n_b|Z|$ (this is also the number of outgoing edges from $Z$ in $E$ with labels from $\Sigma_b$). For $a \in \Sigma_b$, let $E_a$ denote the set of $E_b$-edges labeled $a$ outgoing from the $E$-states in $Z$, and let $Z_a$ denote the set of terminal $E'$-states of these edges; note that $\{E_a\}_{a \in \Sigma_b}$ forms a partition of the set of $E_b$-edges outgoing from $Z$. If $Z_a \not= \emptyset$, then clearly there is an edge $Z \xrightarrow{a} Z_a$ in $H$ and, since $H'$ is a sink, this edge is also contained in $H'$, and, therefore, in $H'_u$. We now claim that any $E'$-state $u \in Z_a$ is accessible in $E'$ (and in $E_b$) by exactly one edge labeled $a$ that starts at $Z$; otherwise if $Z = T_Z(w,v)$, the word $wac$ could be generated in $E$ by two distinct paths which start at $u$ and terminate in $v$, contradicting the losslessness of $E$. Hence, $|E_a| = |Z_a|$ and, so, the entry of $A_{H'_u}c$ corresponding to the $H'$-state $Z$ satisfies

$$(A_{H'_u}c)_Z = \sum_{Y \in V(H')} (A_{H'_u})_{Z,Y}c_Y = \sum_{Y \in V(H')} (A_{H'_u})_{Z,Y}|Y| = \sum_{a \in \Sigma_b} |Z_a| = \sum_{a \in \Sigma_b} |E_a| = n_b|Z| = n_bc_Z,$$

thus proving (6).

(c) Construct from $c$ a nonnegative nonzero integer vector $x$ that satisfies (5). Recalling the definition and notation of a follower set from the beginning of Section I-A, let $x = (x_u)_{u \in V(G)}$ be the nonnegative integer vector whose entries are defined for every $u \in V(G)$ by

$$x_u = \max \left\{ c_Z : Z \in V(H') \text{ and } F_H(Z) \subseteq \mathcal{F}_G(u) \right\}. \hspace{1cm} (7)$$

Also, denote by $Z(u)$ some particular $H'$-state $Z$ for which the maximum is attained in (7); in case the maximum is over an empty set, define $x_u = 0$ and $Z(u) = \emptyset$. Since $S(H') \subseteq S(G)$, we get from [14, Lemma 2.13] that each follower set of an $H'$-state is contained in a follower set of some $G$-state; hence, $x$ is necessarily nonzero.

Next, we show that $x$ satisfies (5). Fix a parity $b \in \{2\}$, and let $u$ be a $G$-state; if $x_u = 0$ then, trivially, $(A_Gx)_u \geq n_bx_u$; so, we assume hereafter that $x_u \not= 0$. Let $\Sigma_b(Z(u))$ denote the set of labels of edges in $H'_b$ outgoing from $Z(u)$, and, for $a \in \Sigma_b(Z(u))$, let $Z_a(u)$ be the terminal state in $H'$ (and in $H'_b$) for an edge labeled $a$ outgoing from the $H'$-state $Z(u)$. Since $F_H(Z(u)) \subseteq \mathcal{F}_G(u)$, there exists an edge labeled $a$ in $G$ from which terminates in some $G$-state $u_a$. Now, since $G$ and $H'$ are both deterministic, we have $F_H(Z(u)) \subseteq \mathcal{F}_G(u_a)$ and, so, by (7) we get that $x_{u_a} \geq c_{Z_a(u)}$. Therefore,

$$(A_Gx)_u \geq \sum_{a \in \Sigma_b(Z(u))} x_{u_a} \geq \sum_{a \in \Sigma_b(Z(u))} c_{Z_a(u)} = (A_{H'_b}c)_{Z(u)} = n_bc_{Z(u)} = n_bx_u,$$

namely, $A_Gx \geq n_bx$.

The following corollary parallels Corollary 1 in [13] (or Theorem 7.2 in [14]).

**Corollary 4.** Let $S$ be an irreducible constraint, presented by an irreducible deterministic graph $G$, and let $n_0$ and $n_1$ be positive integers. Then, for any $(S,n_0,n_1)$-encoder $E$,

$$|V(E)| \geq \min_{x \in X(A_G, A_G, n_0, n_1)} \|x\|_\infty,$$

where $\|(x_u)_u\|_\infty = \max_u x_u$.

**Proof.** Given an $(S,n_0,n_1)$-encoder $E$, construct the vector $x \in X(A_G, A_G, n_0, n_1)$ as in the proof of Theorem 3. Then, by construction, each component of $x$ is a size of a subset of $V(E)$ and, hence, bounds it from below.

The next corollary parallels Theorem 5 in [13] (or Theorem 7.15 in [14]) and is proved in the very same manner.

**Corollary 5.** With $S$, $G$, $n_0$, $n_1$, and $E$ as in Corollary 4,

$$A(E) \geq \log_n \left( \min_{x \in X(A_G, A_G, n_0, n_1)} \|x\|_\infty \right),$$

where $n = \max\{n_0, n_1\}$.

The Franaszek algorithm is a known method for computing approximate eigenvectors [14, §5.2.2]. Figure 6 presents a modification of it for computing a vector in $X(A_0, A_1, n_0, n_1)$, where $A_0$ and $A_1$ are nonnegative integer $k \times k$ matrices (a nonnegative integer $k$-vector $\xi$ is provided as an additional parameter to the algorithm). The modified algorithm can be used to compute the lower bounds of Corollaries 4 and 5, and will turn out to be useful also when designing bi-modal encoders.

By slightly generalizing the proof of validity of the (ordinary) Franaszek algorithm (see [14, §5.2.2]) it follows that the algorithm in Figure 6 returns the largest (componentwise) vector $x \in X(A_0, A_1, n_0, n_1)$ that satisfies $x \leq \xi$; if no such vector exists, then the algorithm returns the all-zero vector.

![Fig. 6. Modified Franaszek algorithm.](image-url)

By analyzing the complexity of Karmarkar’s algorithm [12] (in terms of number of bit operations), one can conceptually infer an upper bound on the smallest possible largest entry of any vector in $X(A_0, A_1, n_0, n_1)$, in terms of $A_0$, $A_1$, $n_0$, and $n_1$ (provided that we first use that algorithm to determine that $X(A_0, A_1, n_0, n_1)$ is nonempty). Equivalently, such an analysis implies an upper bound on the smallest possible integer $\xi > 0$ such that running the algorithm in Figure 6 with $\xi = \xi - 1$ yields a nonzero output $x \in X(A_0, A_1, n_0, n_1)$. It is still open whether there is a more direct way for computing (efficiently) such an upper bound.
III. SUFFICIENT CONDITION

We start proving the “if” part in Theorem 1 by considering two special cases in Sections III-A and III-B. We then turn to the general case in Section III-C.

A. Deterministic encoders

If $\mathcal{X}(A_{G_0}, A_{G_1}, n_0, n_1)$ contains a 0–1 vector, then a subgraph of $G$ is an $(S(G), n_0, n_1)$-encoder with anticipation 0. By Corollary 5, the existence of such a vector is also a necessary condition for having a deterministic $(S(G), n_0, n_1)$-encoder. If, in addition, $G$ has finite memory $\mu$, then the resulting encoder is $(\mu, 0)$-definite and, therefore, $(\mu, 0)$-sliding-block decodable for any tagging.

B. Encoders with anticipation 1 obtained by state splitting

Suppose now that $\mathcal{X}(A_{G_0}, A_{G_1}, n_0, n_1)$ does not contain a 0–1 vector, yet contains a vector $x = (x_u)_{u}$ such that for each $b \in [2]$, an application of one $x$-consistent state splitting round4 to $G_b$ (after deleting all states $u \in G$ with $x_u = 0$) results in an all-1 induced approximate eigenvector. For $b \in [2]$, let $E_b$ be the resulting $(S(G_b), n_0, n_1)$-encoder. Note that each state $u \in V(G)$ is transformed into $x_u$ descendant states in each encoder $E_b$; denote those states by $(u, i)_b$, where $i \in [x_u]$ (the order implied by the index $i$ on the descendant states of a given $u$ can be arbitrary).

Next, construct the following graph $\mathcal{E}$:

$$V(\mathcal{E}) = \{(u, i) : u \in V(G) \text{ and } i \in [x_u]\},$$

and endow $\mathcal{E}$ with an edge $(u, i) \xrightarrow{\sigma} (v, j)$ if and only if for some $b \in [2]$, the encoder $E_b$ contains an edge $(u, i)_b \xrightarrow{\sigma} (v, j)_b$. It follows from the construction that $E_b = E_b$ for $b \in [2]$. In particular, from each $E_b$-state there are $n_b$ outgoing edges with labels from $\Sigma_b$, for each $b \in [2]$. Moreover, it can be readily seen that $F_\mathcal{E}((u, i)) \subseteq F_{G_2}(u)$ for every $u \in V(G)$ and $i \in [x_u]$. Finally, $\mathcal{E}$ has anticipation 1: if a word $w_1w_2$ is generated in $\mathcal{E}$ by a path $\pi$ from state $(u, i) \in V(\mathcal{E})$, then the parent $G$-state $u$ of $(u, i)$ and the symbol $w_1$ uniquely identify the parent $G$-state, $v$, of the terminal state of the first edge in $\pi$, and the symbol $w_2$ then uniquely identifies the particular descendant state $(v, j)$ of $v$ in which that edge terminates.

In summary, $\mathcal{E}$ is an $(S(G), n_0, n_1)$-encoder with anticipation 1. Furthermore, if $G$ has finite memory $\mu$, then $\mathcal{E}$ is $(\mu, 1)$-definite and, therefore, $(\mu, 1)$-sliding-block decodable for any tagging.

Example 5. We consider the 16th power of the $(2, 10)$-RLL constraint, as found in the DVD standard [14, §1.7.3 and Example 5.7]. Let $G$ be the graph presentation of that power, where the states are numbered from 0 to 10, and edges labeled with 16-bit words that end with a run of 0s of length $i \in [11]$ terminate in state $i$. Also, let $\Sigma_0$ (respectively, $\Sigma_1$) be the set of all 16-bit words of even (respectively, odd) parity that satisfy the $(2, 10)$-RLL constraint. Then,


and


Running the algorithm in Figure 6 with $A_0 = A_{G_0}$, $A_1 = A_{G_1}$, and $\xi = 2 \cdot 1$ yields the result

$$x = (1 1 2 2 2 2 1 1 1 1 0)^\top,$$

for any $n_0 \leq 173$ and $n_1 \leq 178$ (running the algorithm with larger values of $n_0$ or $n_1$ yields the all-zero vector). This is also the vector obtained when running the (ordinary) Franaszek algorithm with $A = A_{G_0} + A_{G_1}$, $n = 351$, and $\xi = 2 \cdot 1$. In both $G_0$ and $G_1$ we can merge states 2–5 into 5, states 6–9 into 9, and delete state 10 (see [14, §5.5.1]), resulting in graphs $G_0'$ and $G_1'$ with

$$A_{G_0}' = \begin{pmatrix} 42 & 28 & 45 & 14 \\ 62 & 42 & 67 & 18 \\ 86 & 59 & 90 & 22 \\ 50 & 36 & 59 & 9 \end{pmatrix},$$

and

$$A_{G_1}' = \begin{pmatrix} 41 & 29 & 53 & 8 \\ 60 & 41 & 75 & 14 \\ 78 & 54 & 102 & 20 \\ 47 & 31 & 14 & 16 \end{pmatrix},$$

and the respective $(A_{G_0}', A_{G_1}', n_0=173, n_1=178)$-approximate eigenvector is

$$x' = (1 1 2 1)^\top.$$

Both $G_0'$ and $G_1'$ can be split in one round consistently with $x'$, resulting in the all-1 induced approximate eigenvector and, therefore, in an $(S(G), 173, 178)$-encoder. The out-degree of the encoder that is actually used in the DVD is $2^p = 256$, where the set of input tags consists of all 8-bit tuples. Some of the input tags can be mapped to two possible 16-bit words with different parities5, while the rest are mapped to unique 16-bit words.

4Refer to [14, Ch. 5] for the description of the state-splitting algorithm and for the related terms used here.

5There can be at most $173 + 178 - 256 = 95$ input bytes of this type, but in practice their number is slightly smaller.
C. Construction using the stethering method

The technique used in Section III-B does not seem to generalize easily if the conditions therein—namely, being able to split \( G_0 \) and \( G_1 \) in one round and ending up with an all-1 induced approximate eigenvector—do not hold. In fact, due to the fact that the matrices \( G_0 \) and \( G_1 \) may be reducible, we may get stuck while attempting to split them.

**Example 6.** Let \( G \) be the graph with \( V(G) = \{\alpha, \beta, \gamma\} \) whose even and odd subgraphs, \( G_0 \) and \( G_1 \), are shown in Figures 7 and 8 (note that \( G_1 \) is reducible). All the edges in \( G \) are assumed to have distinct labels. Assuming the ordering \( \alpha < \beta < \gamma \) on the states, the adjacency matrices of the subgraphs are given by

\[
A_{G_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A_{G_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}.
\]

It is easy to see that \( x = (1 \ 2 \ 3)^{T} \) is an eigenvector of both \( A_{G_0} \) and \( A_{G_1} \) associated with the Perron eigenvalue \( n = 2 \). Yet the subgraph \( G_1 \) cannot be split consistently with \( x \). \( \square \)

![Fig. 7. Subgraph \( G_0 \) for Example 6.](image)

![Fig. 8. Subgraph \( G_1 \) for Example 6.](image)

Moreover, in Appendix A we present an example where multiple rounds of state splitting are required, which do end up with an all-1 approximate eigenvector, yet there is no way one can match the descendant states in \( \mathcal{E}_0 \) of a given \( G \)-state with the respective descendant states in \( \mathcal{E}_1 \) while maintaining finite anticipation.

Recognizing that the finite anticipation property is not guaranteed even when the state-splitting algorithm is used (at least in the manner we employed this algorithm in Section III-B), we resort to a more general framework of designing encoders, which includes the state-splitting algorithm and the stethering design method of [3] as special cases (see also [1] and [14, §6.2]). As we see, it will be rather easy to adapt the stethering method to design bi-modal encoders, even though finite anticipation can be guaranteed only under certain conditions.

Next we recall the stethering method, while tailoring it to our setting. Let \( G \) be a deterministic graph and \( \{\Sigma_0, \Sigma_1\} \) be a partition of its label alphabet \( \Sigma \), and let \( x = (x_u)_{u \in V(G)} \) be in \( \mathcal{X}(A_{G_0}, A_{G_1}, n_0, n_1) \). We assume that \( x > 0 \), or else remove the zero-weight states from \( G \) (namely, the states \( u \) for which \( x_u = 0 \)). For \( u \in V(G) \), denote by \( \Sigma_b(u) \) the set of symbols from \( \Sigma_b \) that label edges outgoing from \( u \). For \( u \in V(G) \) and \( a \in \Sigma_b(u) \), denote by \( \tau(u; a) \) the terminal \( G \)-state of the unique edge outgoing from \( u \) with label \( a \).

For \( b \in [2] \) and \( u \in V(G) \), let

\[
\Delta_b(u) = \left\{ (a, j) : a \in \Sigma_b(u) \text{ and } j \in [x_{\tau(u; a)}] \right\}.
\]

Since \( x \in \mathcal{X}(A_{G_0}, n_b) \) we have \( |\Delta_b(u)| = (A_{G_0} x)_u \geq n_b x_u \). Thus, we can partition (a subset of) \( \Delta_b(u) \) into \( x_u \) subsets

\[
\Delta_b^{(i)}(u), \Delta_b^{(1)}(u), \ldots, \Delta_b^{(x_u - 1)}(u),
\]

such that \( |\Delta_b^{(i)}(u)| = n_b \) for each \( i \). In what follows, we fix such a partition.

Next, construct the following graph \( \mathcal{E} \):

\[
V(\mathcal{E}) = \left\{ (u, i) : u \in V(G) \text{ and } i \in [x_u] \right\},
\]

and for each \( b \in [2] \), \( u \in V(G) \), \( i \in [x_u] \), and \((a, j) \in \Delta_b^{(i)}(u)\), we endow \( \mathcal{E} \) with an edge \((u, i) \xrightarrow{a} (\tau(u; a), j)\).

**Proposition 6.** The constructed graph \( \mathcal{E} \) is an \((S(G), n_0, n_1)\)-encoder.

**Proof.** First, by construction, the number of outgoing edges from \((u, i)\) with labels from \( \Sigma_b \) is \( |\Delta_b^{(i)}(u)| = n_b \), for each \( b \in [2] \).

Secondly, let

\[
(u_0, i_0) \xrightarrow{w_1} (u_1, i_1) \xrightarrow{w_2} (u_2, i_2) \xrightarrow{w_3} \ldots \xrightarrow{w_{\ell}} (u_\ell, i_\ell)
\]

be a path in \( \mathcal{E} \). By construction, \( u_{m+1} = \tau(u_m; w_{m+1}) \) for every \( m \in [\ell] \). Hence, \( S(\mathcal{E}) \subseteq S(G) \).

It remains to show that \( \mathcal{E} \) is lossless. Consider the word \( w = w_1 w_2 \ldots w_{\ell} \) generated by the path (9). We show that the knowledge of \( w \), \((u_0, i_0)\), and \((u_\ell, i_\ell)\) uniquely determines the rest of the states along the path. Since \( G \) is deterministic, the component \( u_m \) of each state \((u_m, i_m)\) along the path is uniquely determined. Suppose by induction that \((u_m, i_m)\) has been uniquely determined for every \( m' > m \) and let \( b_{m+1} \) be the parity of \( w_{m+1} \) (i.e., \( w_{m+1} \in \Sigma_{b_{m+1}} \)). Since the subsets in (8) are disjoint for \((u, b) = (u_m, b_{m+1})\), there is a unique index \( i \in [x_{u_m}] \) for which \((w_{m+1}, i_{m+1}) \in \Delta_b^{(i)}(u_m) \); that index must be \( i = i_m \). \( \square \)

The number of states of the constructed encoder \( \mathcal{E} \) (before any possible merging of states) equals the sum, \( \|x\|_1 \), of the entries of \( x \). Thus, with this construction, we can obtain an encoder \( \mathcal{E} \) such that

\[
|V(\mathcal{E})| \leq \min_{x \in \mathcal{X}(A_{G_0}, A_{G_1}, n_0, n_1)} \|x\|_1
\]

(compare with the lower bound of Corollary 4).

In stethering encoders, the subsets in (8) have a particular structure which we describe next. For \( b \in [2] \) and \( u \in V(G) \), assume some ordering on the elements of \( \Sigma_b(u) \). For \( a \in \Sigma_b(u) \), define \( \phi_b(u; a) \) by

\[
\phi_b(u; a) = \sum_{b \in \Sigma_b(u) : b < a} x_{u_b}.
\]
where the sum is zero on an empty set. For \( i \in [x_u] \), let
\[
\Delta_b^{(i)}(u) = \left\{ (a, j) : a \in \Sigma_b(u), j \in [x_{\tau(u,a)}], \right. \\
\text{and } i n_b \leq \phi_b(u;a) + j < (i+1)n_b \}.
\]
In other words, for each \( u, v \in V(G) \), \( i \in [x_u] \), and \( a \in \Sigma_b(u) \), we endow \( E \) with an edge \((u, i) \xrightarrow{a} (v, j)\), if and only if \( v = \tau(u,a) \) and
\[
i n_b \leq \phi_b(u;a) + j < (i+1)n_b.
\]

This construction is illustrated in Figure 9, for a given \( G \)-state \( u \) and parity \( b \in [2] \). The boxes in the top row in the figure represent the “descendants” of state \( u \), namely, the states \((u, i)\), for \( i \in [x_u] \), and the width of each box in the top row is one unit. The outgoing edges from each state \((u, i)\) are shown as downward arrows, along with their labels, and an assignment of input tags, \((b, 0), (b, 1), \ldots (b, n_b-1)\), is shown above the top row. The boxes at the bottom row are \( 1/n_b \) units wide and represent the terminal states of the edges. The respective elements of \( \Delta_b(u) \) are written just below the bottom row, where we have also shown their grouping into the subsets \((8)\) defined by \((10)\). The double vertical lines group the edges according to their labels. So, for example, according to the figure, there is an outgoing edge labeled \( a' \) and tagged by \((b, 0)\) from \((u, x_u-1)\) to \((v', x_v-1)\), and that edge corresponds to the element \((a',x_v-1)\in \Delta_b(u)\).

Serting encoders can have finite (and be sliding-block decodable if \( G \) has finite memory), provided that there is sufficient margin between the target rate \( p : q \) and the maximal coding ratio \( \rho(G,q) \) (as defined in \((2)\)). We demonstrate this next.

Suppose that \( x \in X(\mathcal{A}_{G_0}, \mathcal{A}_{G_1}, n_0+1, n_1+1) \) (namely, we assume even and odd out-degrees larger by 1 than targeted). Using \( x \), we first construct a sterting \((S(G), n_0+1, n_1+1)\)-encoder \( E^* \) and assign the input tags \((b, 0), (b, 1), \ldots (b, n_b)\) to the outgoing edges from each state, as in Figure 9. Then, from \( E^* \) we form a punctured \((S(G), n_0, n_1)\)-encoder \( E \) by deleting all edges in \( E^* \) tagged by either \((0, n_0)\) or \((1, n_1)\).

We have the following result (compare the guaranteed upper bound on \( \mathcal{A}(E) \) to the lower bound in Corollary 5).

**Theorem 7.** Let \( G \) be a deterministic graph and let \( n_0 \) and \( n_1 \) be positive integers such that \( \mathcal{X}(\mathcal{A}_{G_0}, \mathcal{A}_{G_1}, n_0+1, n_1+1) \neq \emptyset \). Then, there is an \((S(G), n_0, n_1)\)-encoder \( E \), obtained by the (punctured) sterting method, such that \( \mathcal{A}(E) \leq a \), where
\[
a = 1 + \min_{x \in X(\mathcal{A}_{G_0}, \mathcal{A}_{G_1}, n_0+1, n_1+1)} \left\{ \frac{\log \min \{n_0+1, |x|_{\infty}\}}{n} \right\}
\]
and \( n = \min\{n_0, n_1\} \). Furthermore, if \( G \) has finite memory \( \mu \), then \( E \) is \((\mu, a)\)-definite, and hence any tagged \((S(G), n_0, n_1)\)-encoder based on \( E \) is \((\mu, a)\)-sliding-block decodable.

**Proof.** The proof is essentially the same as that of Proposition 3 in [3], and we repeat it here (with the required modifications to handle the bi-modal setting) for completeness.

Let \( C_b(u) \) denote Figure 9 drawn for a given \( b \in [2] \) and \( u \in V(G) \). Consider a path
\[
\pi = (u_0, i_0) \xrightarrow{w_1} (u_1, i_1) \xrightarrow{w_2} (u_2, i_2) \xrightarrow{w_3} \ldots
\]
in the encoder \( E^* \) (obtained prior to the puncturing), and let \((b_1, s_1), (b_2, s_2), (b_3, s_3)\ldots \) be the respective sequence of input tags, where \( w_m \in \Sigma_{b_m} \). Envision an array \( B \) which is constructed as follows. Start with the figure \( C_{b_1}(u_0) \), the edges labeled \( w_1 \) in the figure terminate in the descendant states of \( u_1 \), which appear as \( x_{u_1} \) boxes in the bottom row of \( C_{b_1}(u_0) \). Up to down-scaling by a factor of \( n_b \), these boxes are identical to the top row of \( C_{b_2}(u_1) \). So, in \( B \), superimpose a down-scaled copy of \( C_{b_2}(u_1) \) so that its top row coincides with the descendant states of \( u_1 \) in \( C_{b_1}(u_0) \). Proceed in this manner by placing in \( B \) a copy of \( C_{b_2}(u_2) \), down-scaled by a factor of \( n_bn_{b_2} \), so that its top row coincides with the descendant states of \( u_2 \) in the bottom row of the (already inserted) down-scaled copy of \( C_{b_1}(u_1) \). And so on.

The path \( \pi \) can be seen as a vertical line in \( B \) whose abscissa (i.e., the distance from the left margin of \( B \)) has the mixed-base representation \( i_0, s_{b_1}, s_{b_2}, s_{b_3}, \ldots \) where \( i_0 \in [x_u] \) and \( s_{b_m} \in [n_{b_m}+1] \). In other words, that abscissa equals
\[
i_0 + \frac{s_{b_1}}{n_{b_1}+1} + \frac{s_{b_2}}{(n_{b_1}+1)(n_{b_2}+1)} + \ldots
\]
Due to the down-scaling process used to construct \( B \), a decoder can narrow down the uncertainty of that abscissa for each received symbol. Specifically, merely by the knowledge of \( u_0 \) that abscissa must be in the real interval \([0,x_{u_0}]\) (which is the full width of \( B \)). Upon receiving \( w_1 \), the length of that uncertainty (open) interval shrinks to \( x_{u_1}/(n_{b_1}+1) \); then \( w_2 \) reduces it to \( x_{u_2}/((n_{b_1}+1)(n_{b_2}+1)) \), and so forth. Hence, when the length \( \ell \) of the path is such that
\[
\frac{x_{u_\ell}}{\prod_{m=1}^\ell (n_{b_m}+1)} \leq \frac{1}{(n_{b_1}+1)(n_{b_2}+1)},
\]
the length of the uncertainty interval reduces to at most the right-hand side of \((11)\). At this point, the numerator in the expression
\[
\frac{(n_{b_2}+1)s_{b_1}+s_{b_2}}{(n_{b_1}+1)(n_{b_2}+1)}
\]
(for the abscissa point \(0,s_{b_1},s_{b_2}\)) can be determined up to \( \pm 1 \). Yet since the puncturing disallows \( s_{b_2} \) to take the value \( n_{b_2} \), this means that \( s_{b_1} \) is uniquely determined. It is easy to see that \((11)\) is satisfied for \( \ell = a + 1 = 2 + \lfloor \log_{n_{b_1}+1} |x|_{\infty} \rfloor \), where \( n = \min\{n_0, n_1\} \), thereby proving the claimed upper bound on \( \mathcal{A}(E) \). Moreover, if \( G \) has finite memory \( \mu \), then the decoder can recover \( u_0 \) by looking at a window of \( \mu \) past symbols, i.e., \( E \) is \((\mu,a)\)-definite.

Recall that \( n_{\max}(G,q) \) is the largest integer \( n \) for which \((S(G^n), n,n)\)-encoders exist. If we use the punctured sterting method to construct rate \( p : q \) bi-modal encoders, then we need to have \( 2^p+1 \leq n_{\max}(G,q) \). This inequality is satisfied whenever
\[
p \leq \frac{\log_2 n_{\max}(G,q)}{q} - \log_2(1+2^{-p}),
\]
which, in turn, is satisfied whenever
\[
p \leq p(G,q) - \frac{\log_2 e}{2pq}.
\]
(see (2)). We conclude that finite anticipation (and sliding-block decodability when \( G \) has finite memory) can be guaranteed with a rate penalty of (no more than) \( (\log_2 e)/(2^p q) \).

It is still an open problem whether finite anticipation can be guaranteed for any \((n_0, n_1)\) for which \(X(A_{G_0}, A_{G_1}, n_0, n_1) \neq \emptyset\).

### D. Extension to non-partition covers

We refer now to a more general setting where the even subset \(\Sigma_0\) and the odd subset \(\Sigma_1\) of the constraint alphabet \(\Sigma\) are not necessarily disjoint, but their union is still \(\Sigma\) (recall Footnote 2 for an application of this scenario). Thus, given a graph \(G\), the subgraphs \(G_0\) and \(G_1\) may share edges. It turns out that most of our results hold also for this setting. In particular, the necessary condition stated in Theorem 3 holds as is, since the proof of the theorem does not assume that \(\Sigma_0\) and \(\Sigma_1\) are disjoint.

As for the sufficient condition shown in Section III-C, the proof of Proposition 6 holds as long as the partitions (8) satisfy the following condition for every \(u \in V(G), a \in \Sigma_0(u) \cap \Sigma_1(u)\), and \(j \in [x_{\tau(u/n)}]\):

\[
(a, j) \in \Delta_{\text{i}}(u) \cap \Delta_{j}(u) \implies i = j
\]

(12) (this guarantees the uniqueness of the index \(i = i_m\) in the last step of the proof of Proposition 6, even when the parity \(b_{m+1}\) is not uniquely determined by \(w_{m+1}\)). Assuming without loss of generality that \(n_0 \leq n_1\), we can apply the following strategy to guarantee the condition (12). We first select an arbitrary partition (8) for \(b = 0\) where \(|\Delta_{\text{i}}(u)| = n_0\) for each \(i \in [x_u]\). Then, for each \(u \in V(G), a \in \Sigma_0(u) \cap \Sigma_1(u)\), and \(i \in [x_u]\), we let all elements \((a, j) \in \Delta_{\text{i}}(u)\) be also elements of \(\Delta_{j}(u)\). Finally, we fill in each subset \(\Delta_{\text{i}}(u)\) from the remaining elements of \(\Delta_{\text{i}}(u)\) to have \(|\Delta_{\text{i}}(u)| = n_1\).

We note that, in general, the stethering partition (10) (which is used in the proof of Theorem 7) may be inconsistent with the condition (12). Still, we can always get consistency when \(n_0 = n_1\), by selecting the ordering on \(\Sigma_0(u)\) and \(\Sigma_1(u)\) so that the elements in the intersection \(\Sigma_0(u) \cap \Sigma_1(u)\) precede the rest.

### IV. VARIABLE-LENGTH ENCODERS

So far in this work, we considered bi-modal fixed-length encoders at a fixed rate \(p : q\), where all tags have the same length \(p\), and all labels have the same length \(q\) (under the formulation of \((S, n)\)-encoders, these lengths are 1). On the other hand, most ad-hoc constructions for parity-preserving encoders that were proposed have variable length (yet still with a fixed coding ratio). In this section, we show through examples that the flexibility of having variable-length (tags and) labels strictly increases the attainable range of the coding ratios of parity-preserving encoders, compared to the fixed-length case. A more thorough study of parity-preserving variable-length encoders is deferred to a subsequent work [18]. For more on variable-length encoders (in the non-parity-preserving setting), see [2], [4], [5], [6], [7], [8], [14, §6.4].

**Example 7.** Let \(S\) be the constraint over \(\Sigma = \{a, b, c, d\}\) that is presented by the graph \(G\) in Figure 4. Consider the graph \(E\) in Figure 10, which has one state \(\alpha\) and three edges: an edge labeled \(a\) and two edges of length 2, with labels \(bd\) and \(cd\) (namely, the length of an edge is the length of its label). For the purpose of defining the words that can be generated by a variable-length graph such as \(E\), we view each length-\(\ell\) edge as if it were a path of length \(\ell\) (whose edges are connected by additional \(\ell - 1\) dummy states). Doing so, it is easy to see that every word that can be generated by \(E\) can also be generated from state \(\alpha\) in the graph \(G\) of Figure 4. The graph \(E\) is deterministic in the sense that the set of labels is prefix-free: no label is a prefix of any other label. Hence, a word generated by \(E\) uniquely identifies the path that generates it.

We now assign tags over the (base tag) alphabet \(\Upsilon = \{0, 1\}\) to the edges (labels) of \(E\), as shown in Table 1. We get in this manner an encoder that has a coding ratio of 1:1 the coding rate is 1 : 1 when the input tag is 0, and 2 : 2 when the input tag starts with 1. Thus, this encoder is capacity-achieving. Moreover, this tag assignment is parity-preserving with respect to the partition \(\{\Sigma_0, \Sigma_1\}\) defined in (3). In contrast, we showed

<table>
<thead>
<tr>
<th>TAG ASSIGNMENT</th>
<th>ENCODER IN FIGURE 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>← a</td>
</tr>
<tr>
<td>10</td>
<td>← bd</td>
</tr>
<tr>
<td>11</td>
<td>← cd</td>
</tr>
</tbody>
</table>

---

**Fig. 9.** Descendants of a \(G\)-state \(u\) in a subgraph \(E_b\) of a stethering encoder.

**Fig. 10.** Variable-length encoder \(E\) for the constraint presented by Figure 4.
in Example 2 that, for this partition, a coding ratio of 1 cannot be achieved by any bi-modal (and, a fortiori, parity-preserving) fixed-length encoder.

**Example 8.** Considering the same constraint \( S \) as in the previous example, the graph \( E' \) in Figure 11 presents another (untagged) variable-length encoder. The coding rate at state \( \alpha' \) is 3 : 3, as it has eight outgoing edges with labels in \( \Sigma^3 \), and the coding rate at \( \alpha'' \) and at \( \beta \) is 2 : 2, as each state has four outgoing edges labeled from \( \Sigma^2 \); the coding ratio at each state is therefore 1, making \( E' \) capacity-achieving. However, \( E' \) is not deterministic (there are two edges labeled \( bda \) and two labeled \( cda \) outgoing from state \( \alpha' \), two edges labeled \( aa \) outgoing from \( \alpha'' \), and two labeled \( da \) from state \( \beta \)). Nevertheless, \( E' \) has finite anticipation and is therefore lossless: the first symbol of a label uniquely determines the length of the label as well as the initial state, and a label and the first symbol of the next label within a sequence uniquely determine the edge. One possible assignment of tags (over the alphabet \( \Upsilon = \{0,1\} \)) to the edges of \( E' \) is shown in Table II.

![Figure 11. Second variable-length encoder \( E' \) for the constraint presented by Figure 4.](image)

**Table II**

<table>
<thead>
<tr>
<th>State ( \alpha' )</th>
<th>State ( \alpha'' )</th>
<th>State ( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>000, 011</td>
<td>00, 11</td>
<td>01, 10</td>
</tr>
<tr>
<td>101, 110</td>
<td>01</td>
<td>00</td>
</tr>
<tr>
<td>001</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>010</td>
<td>11</td>
<td>00</td>
</tr>
<tr>
<td>100</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>111</td>
<td>00</td>
<td>10</td>
</tr>
</tbody>
</table>

Consider now the partition \( \{\Sigma_0, \Sigma_1\} \) defined in (4). The labels in boldface in Figure 11 are the odd labels with respect to this partition. It can be readily verified that the tag assignment in Table II is parity-preserving. In contrast, recall that we have shown in Example 3 that there is no bi-modal fixed-length encoder at a coding ratio of 1 for this constraint and this partition.

The encoder in Figure 11 can be obtained by first splitting state \( \alpha \) in Figure 4 into states \( \alpha' \) and \( \alpha'' \) which inherit, respectively, the outgoing edge sets \( \{b,c\} \) and \( \{a\} \). The resulting graph is an (ordinary) \((S,2)\)-encoder, yet, for the above partition of \( \Sigma \), the parities of the two outgoing edges from each state are the same. We then replace the outgoing edges from state \( \alpha' \) with the eight paths of length 3 that start at that state; similarly, we replace the outgoing edges from each of the states \( \alpha'' \) and \( \beta \) with the four paths of length 2 that start at the state.

To summarize, for the two different partitions, (3) and (4), of the alphabet \( \Sigma = \{a,b,c,d\} \), Examples 7 and 8 present respective (capacity-achieving) parity-preserving variable-length encoders with a coding ratio of 1: the first encoder is deterministic, while the other is not. In fact, we show in [18] that for the partition (4), one cannot achieve a coding ratio of 1 by any deterministic parity-preserving variable-length encoder (unless one uses a degenerate base tag alphabet containing only even symbols).

On the other hand, there exists such an encoder under some relaxation of the notion of fixed coding ratio, following the encoding model considered in [8]: the tagged encoder \( E^\circ \) in Figure 12 maintains a coding ratio of 1 along each cycle. It is easily seen that while at state \( \alpha \), each outgoing edge is uniquely determined by its first symbol, and while at state \( \beta \), an outgoing edge is uniquely determined by its first two symbols.

**APPENDIX A**

**LIMITATIONS OF STATE SPLITTING**

In contrast to what we have shown in Section III-B, we present here an example where the state-splitting algorithm yields encoders \( E_0 \) and \( E_1 \) with anticipation 3, yet any attempt to match between the descendant states in \( E_0 \) of a given \( G \)-state and the respective descendant states in \( E_1 \) results in an encoder that has no finite anticipation.\(^6\)

Let \( G \) be the graph with \( V(G) = \{\alpha, \beta, \gamma, \delta\} \) whose even and odd subgraphs, \( G_0 \) and \( G_1 \), are shown in Figures 13 and 14: the even-valued (respectively, odd-valued) hexadecimal digits form the set \( \Sigma_0 \) (respectively, \( \Sigma_1 \)). Note that \( G_0 \) and \( G_1 \) are identical graphs except for the edge labeling and for a switch between states \( \gamma \) and \( \delta \).

\(^6\)It is still open whether such an example exists where the anticipation of \( E_0 \) and \( E_1 \) is 2. It is not difficult to construct such an example where some matchings of the descendant states in \( E_0 \) with those in \( E_1 \) of the same \( G \)-state yield an encoder with infinite anticipation.
Assuming the ordering $\alpha < \beta < \gamma < \delta$ on the states, the adjacency matrices of the subgraphs are given by

$$A_{G_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{G_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

and it is easily seen that $x = (1 \ 2 \ 3 \ 3)^T$ is an eigenvector of both matrices associated with the Perron eigenvalue $n = 2$.

We are interested in constructing an $(S(G), 2, 2)$-encoder.

Applying a first round of $x$-consistent state splitting to $G_0$ allows splitting of state $\delta$ (and only that state in $G_0$) into two descendant states: $(\delta, 0)$, which has a weight (i.e., approximate eigenvector entry) of 1 and inherits the outgoing edge labeled $c$, and $(\delta, 1)$, of weight 2, which inherits the outgoing edges labeled $a$ and $e$.

In the second round, state $(\delta, 1)$ can be split into $(\delta, 1, 0)$ and $(\delta, 1, 1)$ (each of weight 1), and state $\gamma$ can be fully split into three descendant states, $(\gamma, \cdot, 0)$, $(\gamma, \cdot, 1)$, and $(\gamma, \cdot, 2)$, each of weight 1. The outgoing picture from the descendant states of $\gamma$ is shown in Figure 15(a), where

$$(\gamma_{0;1}, \gamma_{0;2}, \gamma_{0;3}) = ((\gamma, \cdot, 0), (\gamma, \cdot, 1), (\gamma, \cdot, 2))$$

and

$$(\delta_{0;1}, \delta_{0;2}, \delta_{0;3}) = ((\delta, 0), (\delta, 1, 0), (\delta, 1, 1))$$

(the first subscript in $\gamma_{0;i}$ and $\delta_{0;j}$ indicates that we are splitting the graph $G_0$).

In the third round, state $\beta$ is split into two descendant states, $(\beta, \cdot, \cdot, 0)$ and $(\beta, \cdot, \cdot, 1)$, each of weight 1. At this point, we get an $(S(G_0), 2)$-encoder $E_0$.

**Remark 2.** State $\gamma$ can alternatively be split only partially in the second round, yielding a descendant $(\gamma, \cdot, 0)$ of weight 1 and a descendant $(\gamma, \cdot, 1)$ of weight 2, deferring the splitting of $(\gamma, \cdot, 1)$ to the third round. In this case, the outgoing picture shown in Figure 15(b) is also possible, where now $(\gamma_{0;1}, \gamma_{0;2}, \gamma_{0;3}) = ((\gamma, \cdot, 0), (\gamma, \cdot, 1, 0), (\gamma, \cdot, 1, 1))$.

**Remark 3.** In $E_0$, there are three distinct paths labeled 0 4 from state $\alpha$ to the three descendant states $(\gamma_{0;1}, \gamma_{0;2}, \gamma_{0;3})$ of $\gamma$.

A respective splitting of $G_1$ yields an $(S(G_1), 2)$-encoder $E_1$, in which the possible outgoing pictures from the descendant states of $\delta$ (namely, $\delta_{1;i}$) are shown in Figure 16.

Consider now the $(S(G), 2, 2)$-encoder $E$ obtained by matching the descendant states in $E_0$ of each given $G$-state with the respective descendant states in $E_1$. In particular, we select some bijections

$$\varphi_0 : \{\gamma_{1;1}, \gamma_{1;2}, \gamma_{1;3}\} \rightarrow \{\gamma_{0;1}, \gamma_{0;2}, \gamma_{0;3}\}$$  \hspace{1cm} (13)

and

$$\varphi_1 : \{\delta_{0;1}, \delta_{0;2}, \delta_{0;3}\} \rightarrow \{\delta_{1;1}, \delta_{1;2}, \delta_{1;3}\}.$$  \hspace{1cm} (14)

Next, we show that for every such selection, there is an arbitrarily long word $w$ that can be generated in $E$ from two different descendant states of $\gamma$ (in $E_0$). And since both these
Fig. 16. Possible outgoing pictures from the descendants of state $\delta$ in $E_1$.

states are reachable in $E_0$ from state $\alpha$ by paths labeled 0 4, it will follow that $E$ does not have finite anticipation.

Such a word $w$ will be generated by paths that toggle between $E_0$ and $E_1$ after each symbol. For example, suppose that $\varphi_0(\gamma_{1;i}) = \gamma_{0;i}$ and $\varphi_1(\delta_{0;i}) = \delta_{1;i}$, for $i = 1, 2, 3$. Then the word $6767\ldots$ can be generated by the following two paths:

$$\begin{align*}
\gamma_{0;1} & \rightarrow \delta_{0;1} \rightarrow 7 \rightarrow \gamma_{1;1} \rightarrow \delta_{1;1} \rightarrow 7 \rightarrow \ldots \\
\gamma_{0;2} & \rightarrow \delta_{0;2} \rightarrow 7 \rightarrow \gamma_{1;2} \rightarrow \delta_{1;2} \rightarrow 7 \rightarrow \ldots
\end{align*}$$

and

$$\begin{align*}
\gamma_{0;1} & \rightarrow \delta_{0;1} \rightarrow 7 \rightarrow \gamma_{1;1} \rightarrow \delta_{1;1} \rightarrow 7 \rightarrow \ldots \\
\gamma_{0;2} & \rightarrow \delta_{0;2} \rightarrow 7 \rightarrow \gamma_{1;2} \rightarrow \delta_{1;2} \rightarrow 7 \rightarrow \ldots
\end{align*}$$

**Lemma 8.** For any two bijections $\varphi_0$ and $\varphi_1$ as in (13)–(14) and for any positive integer $\ell$, there are at least two paths of length $\ell$ in $E$ that satisfy the following properties.

(i) The paths generate the same word.

(ii) The paths start at distinct descendants of $\gamma$ in $E_0$.

(iii) The states along each path alternate between descendants of $\gamma$ in $E_0$ and descendants of $\delta$ in $E_1$.

**Proof.** When both subgraphs $G_0$ and $G_1$ are split according to part (b) in Figures 15 and 16, then the word $6767\ldots$ can be generated in $E$ from $\gamma_{0;i}$, $\gamma_{0;2}$, and $\gamma_{0;3}$. Hence, we assume from now on in the proof that at most one of the subgraphs is split according to part (b).

Our proof is by induction on $\ell$. The case $\ell = 1$ is obvious, yet when $G_0$ is split according to Figure 15(b), we will also need to establish the case $\ell = 2$. Let $i, j$ be distinct in $\{1, 2, 3\}$ such that

$$\{\varphi_1(\delta_{0;i}), \varphi_1(\delta_{0;j})\} \neq \{\delta_{1;i}, \delta_{1;j}\}.$$  

By Figure 16, this selection guarantees that $\varphi_1(\delta_{0;i})$ and $\varphi_1(\delta_{0;j})$ share an outgoing label $w'$ in $\{7, 9\}$. Hence, if $G_0$ is split according to part (b), then the word $6w'$ can be generated in $E$ both from $\gamma_{0;i}$ and from $\gamma_{0;j}$.

Turning to the induction step, assume that for some odd positive $\ell$ there exist paths $\pi_1$ and $\pi_2$ that satisfy properties (i)–(iii), and let $w$ be the word generated by both paths; due to the symmetry between Figures 15 and 16, the proof is similar for even $\ell$. Let $\gamma_{0;i}$ (respectively, $\gamma_{0;j}$) be the penultimate state visited along the path $\pi_1$ (respectively, $\pi_2$); note that $i \neq j$, or else $E$ would not be lossless (by Remark 2). We now distinguish between two cases.

**Case 1:** $G_0$ is split according to Figure 15(a). In that figure $\gamma_{0;2}$ and $\gamma_{0;3}$ do not share any outgoing labels and, therefore, $\{i, j\} \neq \{2, 3\}$. Thus, $i$ (say) is 1—and therefore $\pi_1$ terminates in $\delta_{0;1}$—and $j \in \{2, 3\}$, and, without loss of generality, $\pi_2$ terminates in $\delta_{0;2}$; moreover, there exists a path $\pi_2'$ that differs from $\pi_2$ only in that it terminates in $\delta_{0;3}$ instead (and $\pi_2'$ still generates the same word $w$). Now, any descendant state of $\delta$ in $E_1$ shares at least one outgoing label with at least one other descendant state of $\delta$ in $E_1$; hence, $\varphi_1(\delta_{0;1})$ must have a common outgoing label $w' \in \{7, 9\}$ with either $\varphi_1(\delta_{0;2})$ or $\varphi_1(\delta_{0;3})$. Thus, $\pi_1$, as well as either $\pi_2$ or $\pi_2'$, can be extended by an edge of $E_1$ labeled $w'$, to produce two paths of length $\ell+1$ that satisfy properties (i)–(iii) (both paths generating the word $ww'$).

**Case 2:** $G_0$ is split according to Figure 15(b). By our assumption this implies that $G_1$ is split according to Figure 16(a), so we can apply the analysis of Case 1 to length $\ell−1$ (with $G_0$ and $G_1$ switching roles). In particular, there exist paths $\pi_1$, $\pi_2$, and $\pi_3$ of length $\ell−1$, all generating the same word $w$, such that $\pi_1$ and $\pi_2$ start at distinct descendants of $\gamma$ (in $E_0$), $\pi_1$ terminates in $\gamma_{1;1}$, and $\pi_2$ and $\pi_3$ differ only in their last edge: $\pi_2$ terminates in $\gamma_{1;2}$ while $\pi_3$ terminates in $\gamma_{1;3}$. Write $\gamma_{0;i} = \varphi(\gamma_{1;1})$, $\gamma_{0;3} = \varphi(\gamma_{1;2})$, and $\gamma_{0;3} = \varphi(\gamma_{1;3})$.

Similarly to Case 1, $\varphi(\delta_{0;i})$ must have a common outgoing label $w' \in \{7, 9\}$ with either $\varphi(\delta_{0;2})$ or $\varphi(\delta_{0;3})$. Thus, $\pi_1$, as well as either $\pi_2$ or $\pi_3$, can be extended by two edges—the first of $E_0$ labeled 6 or 8 and the second of $E_1$ labeled $w'$—to produce two paths of length $\ell+1$ that satisfy properties (i)–(iii) (both paths generating either the word $w6w'$ or the word $ww8w'$).

**REFERENCES**


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