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# Probabilistic Algorithm for Finding Roots of Linearized Polynomials

**Vitaly Skachek · Ron M. Roth**

**Abstract** A probabilistic algorithm is presented for finding a basis of the root space of a linearized polynomial

$$L(x) = \sum_{i=0}^{t} L_i x^{q^i}$$

over $\mathbb{F}_{q^n}$. The expected time complexity of the presented algorithm is $O(nt)$ operations in $\mathbb{F}_{q^n}$.

**Keywords** Linearized polynomials · Probabilistic algorithms · Root-finding algorithms · Symbolic GCD

## 1 Introduction

Let $q$ be a power of a prime and $n$ be a positive integer. A *linearized polynomial over* $\mathbb{F}_{q^n}$ (with respect to $\mathbb{F}_q$) is a polynomial of the form

$$L(x) \ = \ \sum_{i=0}^{t} L_i x^{q^i} \ ,$$

where $L_i \in \mathbb{F}_{q^n}$.

Linearized polynomials were first investigated by Ore (see [12], [13]). References [9, §3.4] and [11, §4.9] contain rather extensive summaries of the properties of linearized polynomials. In particular, it is known that the mapping $\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$ defined by $x \mapsto L(x)$ is a linear mapping over $\mathbb{F}_q$. Conversely, every

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linear mapping $\mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ over $\mathbb{F}_q$ can be realized as a linearized polynomial of degree less than $q^n$. These properties of linearized polynomials imply that the roots of a given linearized polynomial over $\mathbb{F}_{q^n}$ in any extension field of $\mathbb{F}_q$ form a vector space over $\mathbb{F}_q$. For applications of linearized polynomials to coding theory, see [2], [5], [6], [15], or [16].

One problem that arises in some of those applications is finding a basis over $\mathbb{F}_q$ of the root space in $\mathbb{F}_{q^n}$ of a given linearized polynomial $L(x)$ of degree $q^t < q^n$ over $\mathbb{F}_{q^n}$. To this end, we can first find a representation of the linear mapping $L : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ as an $n \times n$ matrix $A$ over $\mathbb{F}_q$, according to some basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, and then compute a basis of the kernel of $A$ in $\mathbb{F}_q^n$. The fastest algorithms currently known for finding the kernel of an $n \times n$ matrix over $\mathbb{F}_q$ have time complexity which grows at least as $n^{2+\varepsilon}$, where $\varepsilon \approx 0.376$ (see [1] and [4]; this lower bound on the complexity does not take into account the time required to compute the matrix $A$ from $L(x)$).

Alternatively, a basis of the root space of $L(x)$ can be found by an adaptation of Rabin's probabilistic algorithm for root finding of general polynomials over fields of even characteristic [14], taking into account the special structure and properties of linearized polynomials (see also the improved analysis of Ben-Or [3]). It can be shown that such an adapted version of Rabin's algorithm can find *one* nonzero root of a linearized polynomial in expected time complexity of $O(nt^2)$ operations in $\mathbb{F}_{q^n}$.

In this note, we present a fast algorithm for finding a *whole basis* of the root space (in $\mathbb{F}_{q^n}$) of a linearized polynomial over $\mathbb{F}_{q^n}$, in expected time complexity of $O(nt)$ operations in $\mathbb{F}_{q^n}$. Hereafter, by "operations in $\mathbb{F}_{q^n}$" we mean any of the four arithmetic operations—addition, subtraction, multiplication, or division—in $\mathbb{F}_{q^n}$, as well as raising an element to the $q$th power (referred to here as $q$-exponentiation). When we represent $\mathbb{F}_{q^n}$ as a ring of polynomials modulo an irreducible polynomial of degree $n$ over $\mathbb{F}_q$, each of the four arithmetic operations in $\mathbb{F}_{q^n}$ can be implemented using $O(n \log^2 n \log\log n)$ arithmetic operations in $\mathbb{F}_q$ (see [1, §8.3] and [8, §§8.3, 9.1, and 11.1]), and $q$-exponentiation can be implemented by $O(\log q)$ multiplications in $\mathbb{F}_{q^n}$. (Moreover, for a range of values of $q$ and $n$, the representation of $\mathbb{F}_{q^n}$ according to certain normal bases over $\mathbb{F}_q$ allows us to implement all operations in $\mathbb{F}_{q^n}$—including $q$-exponentiation—using $O(n \log^2 n \log\log n)$ arithmetic operations in $\mathbb{F}_q$ [7].) Hence, our algorithm has time complexity of $O(n^2 t \log^2 n \log\log n \log q)$ operations in $\mathbb{F}_q$, making it faster than the aforementioned algorithms whenever $t = o(n^\varepsilon / (\log^2 n \log\log n \log q))$.

## 2 Symbolic GCD

In this section, we summarize several definitions and properties that will be used in the sequel. Most properties can be found in Ore [12, Ch. 1].

Let $L(x)$ and $M(x)$ be linearized polynomials over $\mathbb{F}_{q^n}$. The *symbolic product* of $L(x)$ with $M(x)$ is defined by

$$L(x) \otimes M(x) = L(M(x)) \ .$$

Symbolic product satisfies associativity and distributivity (with respect to ordinary polynomial addition), but in general it does not satisfy commutativity; i.e. $L(x) \otimes M(x)$ and $M(x) \otimes L(x)$ are typically not equal.

Let $L(x)$ and $M(x)$ be linearized polynomials over $\mathbb{F}_{q^n}$ where $M(x) \neq 0$. Using an algorithm akin to ordinary "long division," one can find unique linearized polynomials $Q(x)$ and $R(x)$ over $\mathbb{F}_{q^n}$ such that

$$L(x) = Q(x) \otimes M(x) + R(x) \quad \text{and} \quad \deg R(x) < \deg M(x) . \tag{1}$$

When $R(x) = 0$, we say that $M(x)$ is a *right symbolic divisor* of $L(x)$. The polynomial $M(x)$ is a right symbolic divisor of $L(x)$, if and only if $M(x)$ divides $L(x)$ in the ordinary sense (see [12, p. 561] for the "only if" part; the "if" part is easy to prove).

Let $L(x)$ and $M(x)$ be linearized polynomials over $\mathbb{F}_{q^n}$, not both zero. A *right symbolic greatest common divisor* of $L(x)$ and $M(x)$ is a monic linearized polynomial $G(x)$ over $\mathbb{F}_{q^n}$ of a largest degree such that $G(x)$ is a right symbolic divisor of both $L(x)$ and $M(x)$.

**Proposition 1** [12, Theorem 1] *Let $L(x)$ and $M(x)$ be linearized polynomials over $\mathbb{F}_{q^n}$, not both zero. The right symbolic greatest common divisor of $L(x)$ and $M(x)$ is unique and equals the return value of the algorithm in Figure 1.*

The unique right symbolic greatest common divisor of $L(x)$ and $M(x)$ will be denoted by $\mathrm{rgcd}(L(x), M(x))$.

**Proposition 2** [12, Theorem 2] *Let $L(x)$ and $M(x)$ be linearized polynomials over $\mathbb{F}_{q^n}$, not both zero. Then*

$$\mathrm{rgcd}(L(x), M(x)) = \gcd(L(x), M(x)) ,$$

*where $\gcd(L(x), M(x))$ is the monic (ordinary) greatest common divisor of $L(x)$ and $M(x)$.*

Similarly to (1), for any two linearized polynomials $L(x)$ and $M(x) \neq 0$ over $\mathbb{F}_{q^n}$ there exist unique linearized polynomials $Q(x)$ and $R(x)$ over $\mathbb{F}_{q^n}$ such that

$$L(x) = M(x) \otimes Q(x) + R(x) \quad \text{and} \quad \deg R(x) < \deg M(x) .$$

When $R(x) = 0$, we say that $M(x)$ is a *left symbolic divisor* of $L(x)$. In general, the set of left symbolic divisors of a given linearized polynomial may differ from its set of right symbolic divisors. However, we do have the following result.

**Input:** linearized polynomials $L(x) \neq 0$ and $M(x)$ over $\mathbb{F}_{q^n}$;
$R_{-1}(x) \leftarrow M(x)$; $R_0(x) \leftarrow L(x)$;
for ($i \leftarrow 1$; $R_{i-1}(x) \neq 0$; $i$++)
  $R_i(x) \leftarrow R_{i-2} - Q_i(x) \otimes R_{i-1}(x)$, where $\deg R_i(x) < \deg R_{i-1}(x)$;
normalize $R_{i-2}(x)$ to be monic;
**Output:** $R_{i-2}(x)$.

**Fig. 1** Algorithm for computing $\mathrm{rgcd}(L(x), M(x))$.

**Lemma 3** *A linearized polynomial $M(x)$ over $\mathbb{F}_{q^n}$ is a right symbolic divisor (or an ordinary divisor) of the polynomial $x^{q^n} - x$, if and only if $M(x)$ is a left symbolic divisor of that polynomial.*

*Proof* Starting with the "only if" part, suppose that

$$x^{q^n} - x = P(x) \otimes M(x)$$

for some linearized polynomial $P(x)$ over $\mathbb{F}_{q^n}$. Next write

$$x^{q^n} - x = M(x) \otimes Q(x) + R(x) \tag{2}$$

for two linearized polynomials $Q(x)$ and $R(x)$ such that $\deg R(x) < \deg M(x)$. Computing the symbolic product of $P(x)$ with both sides of (2) we obtain

$$\begin{aligned} P(x) \otimes (x^{q^n} - x) &= P(x) \otimes M(x) \otimes Q(x) + P(x) \otimes R(x) \\ &= (x^{q^n} - x) \otimes Q(x) + P(x) \otimes R(x) \ . \end{aligned}$$

Now, the two polynomials $P(x) \otimes (x^{q^n} - x)$ and $(x^{q^n} - x) \otimes Q(x)$ vanish at each element of $\mathbb{F}_{q^n}$; therefore, so must $P(x) \otimes R(x)$. On the other hand, since $\deg R(x) < \deg M(x)$, we have

$$\deg(P(x) \otimes R(x)) < \deg(P(x) \otimes M(x)) = \deg(x^{q^n} - x) = q^n \ .$$

Hence, $P(x) \otimes R(x) = 0$ and, so, $R(x) = 0$.

The proof of the "if" part is similar. □

## 3 Finding roots of linearized polynomials

Figure 2 presents a probabilistic algorithm for finding a basis over $\mathbb{F}_q$ of the roots in $\mathbb{F}_{q^n}$ of a given linearized polynomial $L(x)$ over $\mathbb{F}_{q^n}$. Assuming that $L(x) \neq 0$, we let $t$ denote $\log_q \deg L(x)$.

By Proposition 2 we have that the computed linearized polynomial $G(x)$ in Figure 2 equals $\gcd(L(x), x^{q^n} - x)$. So, the roots of $G(x)$ in $\mathbb{F}_{q^n}$ are the

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<u>**Input:**</u> linearized polynomial $L(x)$ over $\mathbb{F}_{q^n}$;
$G(x) \leftarrow \text{rgcd}(L(x), x^{q^n} - x)$; /* use the algorithm in Figure 1 */
denote $r = \log_q \deg G(x)$;
compute a linearized polynomial $H(x)$ such that $x^{q^n} - x = G(x) \otimes H(x)$;
for ($j \leftarrow 0$; $j < r$; $j$++) {
  do {
    select at random an element $z_j \in \mathbb{F}_{q^n}$;
  }
  while $H(z_j)$ is in the linear span of $\{H(z_\ell)\}_{\ell=0}^{j-1}$ over $\mathbb{F}_q$;
}
<u>**Output:**</u> basis elements $H(z_0), H(z_1), \ldots, H(z_{r-1})$.

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**Fig. 2** Algorithm for finding a basis of the root space of $L(x)$ in $\mathbb{F}_{q^n}$.

roots of $L(x)$ in that field, and the dimension of the linear space of these roots is $r = \log_q \deg G(x)$. Lemma 3 implies that $G(x)$ is also a left symbolic divisor of $x^{q^n} - x$ and, thus, the polynomial $H(x)$ in Figure 2 is well defined.

Given that the algorithm in Figure 2 halts, it is rather straightforward to see that it returns a basis of the root space of $G(x)$ and, hence, of $L(x)$. The rest of this section is devoted to analyzing the time complexity of the algorithm.

**Lemma 4** *The polynomial $G(x)$ in Figure 2 can be computed using less than $3(n+1)(t+1)$ operations in $\mathbb{F}_{q^n}$.*

*Proof* When $t \le n$ (the typical case), we apply the algorithm in Figure 1 to $R_{-1}(x) = x^{q^n} - x$ and $R_0(x) = L(x)$. Otherwise, we switch the roles of $R_{-1}(x)$ and $R_0(x)$.

Denote by $\nu$ the largest value of $i$ in Figure 1 for which $R_i(x) \ne 0$ and, for $i = -1, 0, 1, \ldots, \nu$, let $\tau_i$ stand for $(\log_q \deg R_i) + 1$. Using symbolic long division to implement each iteration in the main loop in Figure 1, iteration $i$ requires less than

$$3(\tau_{i-2} - \tau_{i-1} + 1)\tau_{i-1}$$

operations in $\mathbb{F}_{q^n}$. Hence, the overall number of operations in $\mathbb{F}_{q^n}$ that are required to compute $G(x)$ (without applying the last normalization step in Figure 1) is less than three times the value of

$$\begin{aligned}
&\sum_{i=1}^{\nu+1} (\tau_{i-2} - \tau_{i-1} + 1)\tau_{i-1} \\
&\qquad \le (\tau_{-1} - \tau_0 + 1)\tau_0 + \sum_{i=2}^{\nu+1} \Big( \tau_{i-2}(\tau_{i-2} - 1) - \tau_{i-1}(\tau_{i-1} - 1) \Big) \\
&\qquad \le \tau_{-1}\tau_0 \\
&\qquad = (n+1)(t+1) \ .
\end{aligned}$$

The result follows. □

**Lemma 5** *The polynomial $H(x)$ in Figure 2 can be computed using less than $3(n-r+1)(r+1)$ operations in $\mathbb{F}_{q^n}$ (where $r = \log_q \deg G(x)$).*

*Proof* Compute $H(x)$ by symbolic long division. □

**Lemma 6** *Given the polynomial $H(x)$, the expected number of operations in $\mathbb{F}_{q^n}$ needed to compute the basis elements $H(z_0), H(z_1), \ldots, H(z_{r-1})$ in Figure 2 is less than $3n(r+2)$.*

*Proof* In iteration $j$ (which selects $z_j$), the values

$$H(z_0), H(z_1), \ldots, H(z_{j-1})$$

are linearly independent over $\mathbb{F}_q$. Since $\deg H(x) = q^{n-r}$ and $H(x)$ (being a right symbolic divisor of $x^{q^n} - x$) divides $x^{q^n} - x$ in the ordinary sense, it follows that the size of the kernel of the linear mapping $x \mapsto H(x)$ is $q^{n-r}$. Therefore, when $z_j$ is randomly selected from $\mathbb{F}_{q^n}$, the probability that $H(z_j)$ is not in the linear span of $\{H(z_\ell)\}_{\ell=0}^{j-1}$ equals

$$\frac{q^n - q^{n-r+j}}{q^n} = 1 - q^{j-r} ,$$

and the expected number of random selections until $H(z_j)$ satisfies this property is

$$\frac{1}{1-q^{j-r}} = 1 + \frac{1}{q^{r-j}-1} \le 2 .$$

Summing over $j$, the expected overall number of elements that are randomly selected in Figure 2 is

$$\sum_{j=0}^{r-1} \Big( 1 + \frac{1}{q^{r-j}-1} \Big) < r + 2 .$$

Now, for each selected element $z_j$, we compute $H(z_j)$ using at most $3(n-r)+1$ operations in $\mathbb{F}_{q^n}$. Then, we check whether $H(z_j)$ is in the linear span of the set $\{H(z_\ell)\}_{\ell=0}^{j-1}$. To this end, we assume that the $j$ elements of this set have been written as row vectors in $\mathbb{F}_q^n$ thereby forming a $j \times n$ matrix over $\mathbb{F}_q$, and that this matrix has been brought to an upper-echelon form; we then append $H(z_j)$ as a $(j{+}1)$st row to that matrix. Checking whether that row is linearly dependent of the previous rows can be done by Gaussian elimination, which, in turn, requires no more than $2j+1$ operations in $\mathbb{F}_{q^n}$ (specifically, each addition of rows in the matrix amounts to one addition in $\mathbb{F}_{q^n}$, and each multiplication by a scalar from $\mathbb{F}_q$ is over-counted as one multiplication in $\mathbb{F}_{q^n}$). Hence, the overall expected number of operations in $\mathbb{F}_{q^n}$ needed to find $H(z_0), H(z_1), \ldots, H(z_{r-1})$ is at most

$$\begin{aligned} \sum_{j=0}^{r-1} \Big( 3(n-r) + 1 + (2j+1) \Big) \left( 1 + \frac{1}{q^{r-j}-1} \right) & \\ &< 3(n-r)(r+2) + 4 \sum_{j=0}^{r-1} (j+1) \\ &< 3n(r+2) . \end{aligned}$$

□

Summing up the results of Lemmas 4, 5, and 6, we conclude that the overall number of operations in $\mathbb{F}_{q^n}$ of the algorithm in Figure 2 is less than $9(n+1)(t+2)$.

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