Independent Sets in Regular Hypergraphs and Multi-Dimensional Runlength-Limited Constraints*

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Abstract

Let G be a t-uniform s-regular linear hypergraph with r vertices. It is shown that the number of independent sets $I(G)$ in G satisfies

$$\log_2 I(G) \le \frac{r}{t}\left(1 + O\left(\frac{\log^2(ts)}{s}\right)\right).$$

This leads to an improvement of a previous bound by Alon obtained for $t = 2$ (i.e., for regular ordinary graphs). It is also shown that for the Hamming graph $\mathcal{H}(n,q)$ (with vertices consisting of all n-tuples over an alphabet of size q and edges connecting pairs of vertices with Hamming distance 1),

$$\frac{\log_2 I(\mathcal{H}(n,q))}{q^n} = \frac{1}{q} + O\left(\frac{\log^2(qn)}{qn}\right).$$

The latter result is then applied to show that the Shannon capacity of the n-dimensional (d,∞)-runlength-limited (RLL) constraint converges to $1/(d+1)$ as n goes to infinity.

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1 Introduction

For a hypergraph G, let V_G and E_G, respectively, denote the set of vertices and set of hyperedges of G, where $E_G \subseteq \{e \subseteq V_G : |e| \geq 2\}$. For a vertex v in V_G let $N_G(v)$ denote the set of vertices that are adjacent to v in G, namely,

$$N_G(v) = \Big\{v' \in V_G \setminus \{v\} : \{v, v'\} \subseteq e \text{ for some } e \in E_G\Big\},$$

and let $\delta_G(v) = |N_G(v)|$ be the degree of v in G. An independent set in G is a subset $T \subseteq V_G$ such that $|e \cap T| \leq 1$ for all $e \in E_G$. The number of independent sets in G will be denoted by $I(G)$.

A hypergraph G is *t-uniform* if each hyperedge contains t vertices, and is called s-regular if each vertex is contained in s hyperedges. If the intersection of any two hyperedges of G contains at most one vertex then G is said to be *linear*.

The following theorem is the main result of this paper.

Theorem 1.1 *Let G be a t-uniform s-regular linear hypergraph with r vertices. The number of independent sets $I(G)$ in G satisfies*

$$\log_2 I(G) \leq \frac{r}{t}\left(1 + O\left(\frac{\log^2(ts)}{s}\right)\right).$$

The proof of Theorem 1.1 is given in Section 2, and in Section 3 we present a generalization of Theorem 1.1 to uniform linear hypergraphs that are not necessarily regular.

We next present several applications of Theorem 1.1.

1.1 Regular graphs

For the special case of (undirected) regular ordinary graphs, Theorem 1.1 takes the following form.

Theorem 1.2 *For an s-regular graph G with r vertices,*

$$\frac{\log_2 I(G)}{r} \leq \frac{1}{2} + O\left(\frac{\log^2 s}{s}\right). \tag{1}$$

Theorem 1.2 improves on the error term, $O(s^{0.1})$, which was previously obtained by Alon [1] (as shown by Kahn [6], the error term can be further improved to $O(1/s)$ when the s-regular graph G is bipartite). Unfortunately, (1) is not tight for the widely conjectured worst-case graph consisting of a disjoint union of complete bipartite graphs with degree s [1], [6]. Thus, there is still room for improvement.

1.2 Hamming graphs

Let $\mathcal{H}(n,q)$ denote the Hamming graph whose vertices are all indices $\mathbf{j} \in \{0,1,\ldots,q-1\}^n$ and two vertices are connected by an edge if and only if they are at Hamming distance 1 apart, i.e., the vertices differ on exactly one coordinate.

The number, $I(\mathcal{H}(n,q))$, of independent sets in $\mathcal{H}(n,q)$ has received some attention in the literature ($I(\mathcal{H}(n,q))$ is also the number of codes of length n and minimum Hamming distance ≥ 2 over an alphabet of size q). The case $q=2$ is of particular interest, and $\mathcal{H}(n,2)$ is more commonly known as the binary Hamming hypercube. The strongest result for $q=2$ is due to Korshunov and Sapozhenko [8] (see also [13]), who show that

$$I(\mathcal{H}(n,2)) \sim 2\sqrt{\mathrm{e}}\, 2^{2^{n-1}},$$

where e is the base of natural logarithms; it readily follows that $2^{-n}\log_2 I(\mathcal{H}(n,2)) = 1/2 + O(2^{-n})$.

As for general q, we have

$$\frac{\log_2 I(\mathcal{H}(n,q))}{q^n} \geq \frac{1}{q}\,, \tag{2}$$

since every subset of $\{\mathbf{j} = (j_1, j_2, \ldots, j_n) : j_1 + j_2 + \ldots + j_n \equiv 0 \pmod q\}$ is an independent set in $\mathcal{H}(n,q)$.

Little seems to be known about how tight the lower bound (2) is when $q > 2$. Numerical computations of $I(\mathcal{H}(n,q))$ for $q = 2,3,4$ and small n have been carried out [16]. We are not aware of any asymptotic analysis of $I(\mathcal{H}(n,q))$ for $q > 2$ beyond what we derive here. Specifically, we note that a subset of the Hamming graph $\mathcal{H}(n,q)$ is an independent set if and only if it is also an independent set in the q-uniform, n-regular, linear hypergraph with the same vertex set as $\mathcal{H}(n,q)$ and with hyperedges being the subsets of vertices of $\mathcal{H}(n,q)$ that agree in all but one component. Hence, by setting $r = q^n$, $s = n$, and $t = q$ in Theorem 1.1 we obtain the following result.

Theorem 1.3 *The number of independent sets in the Hamming graph $\mathcal{H}(n,q)$ satisfies*

$$\frac{\log_2 I(\mathcal{H}(n,q))}{q^n} = \frac{1}{q} + O\left(\frac{\log^2(qn)}{qn}\right),$$

for all q.

1.3 Multi-dimensional runlength-limited constraints

For any n-tuple of positive integers $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ let Γ be an n-dimensional $m_1 \times m_2 \times \ldots \times m_n$ binary array whose entries are indexed by n-tuples of integers

$$\mathbf{j} \in \{0, 1, \ldots, m_1 - 1\} \times \{0, 1, \ldots, m_2 - 1\} \times \ldots \times \{0, 1, \ldots, m_n - 1\}.$$

We say that Γ satisfies the (d, ∞)-runlength-limited (RLL) constraint if and only if for any two indices $\mathbf{j}$ and $\mathbf{j}'$ that differ in only one component and differ by less than $d+1$ in that component, either $\Gamma(\mathbf{j}) = 0$ or $\Gamma(\mathbf{j}') = 0$. That is, every one-dimensional sub-array of Γ satisfies the one-dimensional (d, ∞)-RLL constraint. Let $\mathcal{A}(n, d, \mathbf{m})$ be the set of all such arrays. The Shannon capacity of the n-dimensional (d, ∞)-RLL constraint is defined by

$$\mathcal{C}(n, d) = \lim_{i \to \infty} \frac{\log_2 |\mathcal{A}(n, d, \mathbf{m}^{(i)})|}{\prod_{\ell=1}^{n} m_\ell^{(i)}} \tag{3}$$

$$= \inf_{\mathbf{m}} \frac{\log_2 |\mathcal{A}(n, d, \mathbf{m})|}{\prod_{\ell=1}^{n} m_\ell}, \tag{4}$$

where $\mathbf{m}^{(i)} = (m_1^{(i)}, m_2^{(i)}, \ldots, m_n^{(i)})$ is any sequence of n-tuples of integers satisfying $\min_\ell m_\ell^{(i)} \to \infty$. That the right-hand side of (3) is independent of how the limit is taken and coincides with (4) follows from sub-additivity arguments; see [5], [7].

The value $\mathcal{C}(n, d)$ equals the largest coding rate of any encoder (i.e., one-to-one mapping) from the set of finite unconstrained binary sequences into the set of (d, ∞)-RLL constrained arrays [15]. One-dimensional RLL constraints are common in magnetic and optical recording channels [9], [10], [14]. The ongoing practical interest in using multi-dimensional recording media (see, for example [4] and [17]) provides the motivation for studying the values of $\mathcal{C}(n, d)$ for n greater than 1.

The following facts about $\mathcal{C}(n, d)$ are known:

1. $\mathcal{C}(1, d) = \log_2 \alpha_d$, where α_d is the positive real root of the polynomial $x^{d+1} - x^d - 1$ [14, p. 65], [15].
2. $\mathcal{C}(2, d) \sim (\log_2 d)/d$ (namely, $\lim_{d \to \infty} \mathcal{C}(2, d) \cdot (d/\log_2 d) = 1$) [7].
3. $0.5878911617 \le \mathcal{C}(2, 1) \le 0.5878911619$ [3], [12], [17].
4. $0.5225 \le \mathcal{C}(3, 1) \le 0.5269$ [12].
5. $\mathcal{C}(n, d) \ge 1/(d+1)$ for all n [5], [7]. This follows by further constraining the 1's in Γ to have indices $j_1, j_2, \ldots, j_n$ satisfying $j_1 + j_2 + \ldots + j_n \equiv 0 \pmod{d+1}$.

The last fact, together with the simple observation that $\mathcal{C}(n, d)$ is decreasing in n for fixed d (implied by the infimum-based specification of $\mathcal{C}(n, d)$ in (4)), raises the possibility that $\mathcal{C}(n, d)$ decreases with n all the way down to $1/(d+1)$. We next show that this is indeed the case.

Let $\mathcal{H}(n,q)$ be the Hamming graph as defined in Section 1.2 and denote by $\mathbf{1}$ the n-tuple consisting of all 1's. It is not hard to see that the set of locations of 1's in any array in $\mathcal{A}(n,d,(d+1)\mathbf{1})$ corresponds to an independent set in the graph $\mathcal{H}(n,d+1)$. The reverse is also true. Hence,

$$|\mathcal{A}(n,d,(d+1)\mathbf{1})| = I(\mathcal{H}(n,d+1)).$$

On the other hand, we also have the upper bound

$$\mathcal{C}(n,d) \le \frac{\log_2 |\mathcal{A}(n,d,(d+1)\mathbf{1})|}{(d+1)^n}.$$

By Theorem 1.3 we thus get the next result.

Theorem 1.4

$$\lim_{n\to\infty} \mathcal{C}(n,d) = \frac{1}{d+1}.$$

2 Independent sets in uniform, regular, linear hypergraphs

In this section we prove Theorem 1.1. Given a hypergraph G and a subset $Y \subseteq V_G$, let G_Y be the induced (i.e., maximally connected) sub-hypergraph of G on the vertices Y, that is,

$$V_{G_Y} = Y \quad \text{and} \quad E_{G_Y} = \Big\{ e \cap Y : e \in E_G, |e \cap Y| \ge 2 \Big\}.$$

Let $\mathcal{S}_i(G)$ be the set of all induced sub-hypergraphs of G on i vertices, namely,

$$\mathcal{S}_i(G) = \{G_Y : Y \subseteq V_G, |Y| = i\}.$$

Define $f_i(G)$ as

$$f_i(G) = \max_{H\in\mathcal{S}_i(G)} I(H). \tag{5}$$

Note that $f_1(G) = 2$, $f_{|V_G|}(G) = I(G)$, $f_i(G) \ge f_{i-1}(G)$ for $1 < i \le |V_G|$, and

$$f_i(G) \le 2^i. \tag{6}$$

We also define $f_0(G) = 1$ as standing for the empty independent set in an 'empty' sub-hypergraph. Let $\mathcal{S}_i^*(G)$ denote the subset of sub-hypergraphs in $\mathcal{S}_i(G)$ that achieve the maximum in (5). We then have the following simple lemma.

Lemma 2.1 *Given a hypergraph G and an integer i in the range $1 \le i \le |V_G|$, let Δ be a nonnegative integer that satisfies $\Delta \le \delta_H(v)$ for some vertex v of some sub-hypergraph $H \in \mathcal{S}_i^*(G)$. Then*

$$f_i(G) \le f_{i-1}(G) + f_{i-\Delta-1}(G). \tag{7}$$

Proof. For any sub-hypergraph $H \in \mathcal{S}_i^*(G)$ and any vertex $v \in V_H$, the number of independent sets $I(H) = f_i(G)$ is equal to the sum of the number of independent sets that contain v and the number of independent sets that do not contain v. The latter is

$$I(H_{V_H \setminus \{v\}}) \leq f_{i-1}(G)$$

and the former is

$$I(H_{V_H \setminus (\{v\} \cup N_H(v))}) \leq f_{i-\delta_H(v)-1}(G).$$

The lemma follows from the fact that $f_i(G)$ is non-decreasing in i. □

The idea behind the proof of Theorem 1.1 is to start the recursion (7) with the bound $f_{i_0}(G) \leq 2^{i_0}$ for some i_0 and then proceed by bounding the result of iterating the recursion (7) up to $i = |V_G|$. The key to obtaining a good final bound is, for each i, to choose H and v to make Δ in (7) as large as possible. The extent to which this can be done depends on the structure of G.

Specializing to uniform, regular, linear hypergraphs, the following lemma provides a lower bound on the largest possible choice for Δ, for each i.

Lemma 2.2 *Let G be a t-uniform, s-regular, linear hypergraph with r vertices. Then for every $H \in \mathcal{S}_i(G)$*

$$\max_{v \in H} \delta_H(v) \geq \max\left\{ \left\lceil s\left(\frac{ti}{r} - 1\right)\right\rceil, 0 \right\}. \tag{8}$$

Proof. Fix a sub-hypergraph $H \in \mathcal{S}_i(G)$. We prove the lemma by counting ordered pairs of adjacent vertices in V_H in two different ways. Let

$$P = \Big\{(v, v') \in V_H \times V_H : v \neq v' \text{ and } \{v, v'\} \subseteq e \text{ for some } e \in E_G\Big\},$$

and for every $e \in E_G$ let $\beta_e = |e \cap V_H|$. Then $|P| = \sum_{e \in E_G} \beta_e(\beta_e - 1)$; that is, for each hyperedge in G we count the number of ordered pairs of elements of V_H in that hyperedge and sum this over all hyperedges. By the linearity of G each ordered pair is counted only once. Further, $\sum_{e \in E_G} \beta_e = si$ since each vertex $v \in V_H$ contributes to the sum for precisely the s hyperedges that contain it.

Since the function $(\beta_e)_{e \in E_G} \mapsto \sum_{e \in E_G} \beta_e(\beta_e - 1)$ is Schur convex [11] in the variables β_e, its minimum value subject to the constraint $\sum_{e \in E_G} \beta_e = si$ is achieved when β_e is constant-valued.[1]

[1]We can obtain a tighter bound on $\max_{v \in H} \delta_H(v)$ by not ignoring the fact that β_e is integer-valued. In this case, the minimizing β_e takes on at most two values that differ by 1. The resulting bound, however, is more complicated and only slightly improves our bounds on the asymptotic number of independent sets.

And, since $|E_G| = rs/t$, the minimizing β_e is $si/(rs/t) = ti/r$. Therefore,

$$\begin{aligned} |P| &\geq \min_{\beta_e} \sum_{e \in E_G} \beta_e(\beta_e - 1) \\ &= \frac{rs}{t}\frac{ti}{r}\left(\frac{ti}{r} - 1\right) \\ &= si\left(\frac{ti}{r} - 1\right). \end{aligned}$$

On the other hand, letting $\Delta = \max_{v \in H} \delta_H(v)$, we clearly have $|P| \leq \Delta|V_H| = \Delta i$. Combining the two bounds on $|P|$ and dividing by i gives (8). □

We also need the following two elementary propositions.

Proposition 2.3 *The equation $x^{m+1} = x^m + 1$ has only one positive real solution α_m, which is decreasing in m. Further, $\alpha_m \leq m^{1/m}$ for $m \geq 3$.*

Proof. Write the equation as $x^m(x-1) = 1$. The left-hand side is non-positive for x in the range $0 \leq x \leq 1$ and monotonically increasing for $x \geq 1$, implying that there is only one solution $\alpha_m > 1$. By definition $\alpha_m^m(\alpha_m - 1) = 1$ so that $\alpha_m^{m+1}(\alpha_m - 1) > 1$, implying, in turn, that $\alpha_{m+1} < \alpha_m$. Finally, for every $m \geq 3$ we have

$$x^m(x-1)|_{x=m^{1/m}} = m\left(m^{1/m} - 1\right) = m\left(\mathrm{e}^{(\log_\mathrm{e} m)/m} - 1\right) \geq m \cdot \frac{\log_\mathrm{e} m}{m} = \log_\mathrm{e} m > 1,$$

thus implying that $\alpha_m \leq m^{1/m}$. □

Proposition 2.4 *Let $0 = m_0 < m_1 < \ldots < m_\ell$ and $0 = i_{-1} < i_0 < i_1 < \ldots < i_\ell$ be integers such that $i_{j-1} \geq m_j$ for $j = 1, 2, \ldots, \ell$, and suppose that the integer sequence $(f_i)_{i=0}^{i_\ell}$ satisfies*

$$f_i \leq f_{i-1} + f_{i-m_j-1}, \qquad 1 \leq i \leq i_\ell,$$

where $j = j(i)$ is the unique index such that $(m_j \leq)\ i_{j-1} < i \leq i_j$. Let the real sequence $(g_i)_{i=0}^{i_\ell}$ be defined recursively by $g_0 = f_0$ and

$$g_i = \alpha_{m_j} g_{i-1}, \qquad 1 \leq i \leq i_\ell,$$

where j is such that $i_{j-1} < i \leq i_j$ and α_{m_j} is the positive real solution of $x^{m_j+1} = x^{m_j} + 1$. Then $f_i \leq g_i$ for all $0 \leq i \leq i_\ell$.

Proof. We prove by induction on i, where the induction base $i = 0$ is obvious. Turning to the induction step, suppose that $f_{i'} \leq g_{i'}$ holds for all $0 \leq i' < i$ and let j be such that

$i_{j-1} < i \le i_j$. Then

$$\begin{aligned} f_i &\le f_{i-1} + f_{i-m_j-1} \\ &\le g_{i-1} + g_{i-m_j-1} && (9) \\ &\le (1+\alpha_{m_j}^{-m_j}) g_{i-1} && (10) \\ &= \alpha_{m_j} g_{i-1} && (11) \\ &= g_i, \end{aligned}$$

where (9) follows from the induction hypothesis, (10) follows from the definition of g_i and the fact that α_{m_j} is decreasing in j (Proposition 2.3), and (11) follows from the definition of α_m.□

Proof of Theorem 1.1. Let $\Delta(i)$ equal the right-hand side of (8). For $j = 0, 1, \ldots, \ell$, let $m_0 < m_1 < \ldots < m_\ell$ be the values taken on by $\Delta(i)$ as i increases from 0 to r; clearly, $m_0 = 0$ and $m_\ell = s(t-1)$. Denote by i_j the largest i for which $\Delta(i) = m_j$. Thus,

$$i_j = \left\lfloor \left(\frac{m_j}{s} + 1 \right) \frac{r}{t} \right\rfloor \tag{12}$$

and, in particular, $i_0 = \lfloor r/t \rfloor$ and $i_\ell = r$. Since $|N_H(v)| \le i - 1$ for every vertex v in every $H \in \mathcal{S}_i(G)$ and since $|N_H(v)| \ge m_j$ for some v when $i = i_{j-1}+1$, we have $i_{j-1} \ge m_j$. Therefore, by Lemma 2.1, the sequence $(f_i(G))_{i=0}^{i_\ell}$ with the integers m_j and i_j satisfy the assumptions of Proposition 2.4. Hence,

$$\begin{aligned} \log_2 f_r(G) &= \log_2 f_{i_\ell}(G) \\ &\le \log_2 f_{i_0}(G) + \sum_{j=1}^{\ell} (i_j - i_{j-1}) \log_2 \alpha_{m_j} \\ &\le i_0 + \sum_{j=1}^{\ell} (i_j - i_{j-1}) \log_2 \alpha_{m_j}, && (13) \end{aligned}$$

where α_{m_j} is the positive real solution of $x^{m_j+1} = x^{m_j} + 1$ and (13) follows from (6). Incorporating $i_j - i_{j-1} \le (m_j - m_{j-1})r/(ts) + 1$ (from (12)) and $i_0 \le r/t$ into (13) yields

$$\begin{aligned} \log_2 f_r(G) &\le \frac{r}{t} + \sum_{j=1}^{\ell} \left((m_j - m_{j-1}) \frac{r}{ts} + 1 \right) \log_2 \alpha_{m_j} \\ &\le \frac{r}{t} + \sum_{m=1}^{m_\ell} \left(\frac{r}{ts} + 1 \right) \log_2 \alpha_m && (14) \\ &\le \frac{r}{t} + \left(\frac{r}{ts} + 1 \right) \left(2 + \sum_{m=3}^{m_\ell} \frac{\log_2 m}{m} \right) && (15) \\ &\le \frac{r}{t} + \left(\frac{r}{ts} + 1 \right) \left(2 + \frac{\log_2^2 (s(t-1))}{\log_2 \mathrm{e}} \right) && (16) \\ &\le \frac{r}{t} \left(1 + O\left(\frac{\log^2 (ts)}{s} \right) \right), && (17) \end{aligned}$$

where (14) follows since α_m is decreasing in m, (15) follows since $\alpha_2 < \alpha_1 < 2$ and $\log_2 \alpha_m \leq (1/m)\log_2 m$ for $m \geq 3$ (Proposition 2.3), and (16) follows from the fact that $\sum_{m=3}^{m_\ell} 1/m \leq \log_e m_\ell = \log_e s(t-1)$. The bound $r \geq m_\ell = s(t-1)$ justifies (17). The proof is completed by noting that $I(G) = f_r(G)$. □

3 Non-regular hypergraphs

In this section, we generalize Theorem 1.1 to uniform linear hypergraphs that are not necessarily regular.

Given a hypergraph G, let $v_1, v_2, \ldots, v_{|V_G|}$ be a labeling of the vertices of G satisfying $\delta_G(v_1) \leq \delta_G(v_2) \leq \ldots \leq \delta_G(v_{|V_G|})$. For $i = 1, 2, \ldots, |V_G|$ define

$$\sigma_G(i) = \frac{1}{i}\sum_{j=1}^{i} \delta_G(v_j).$$

That is, $\sigma_G(i)$ is the average degree among the i vertices with smallest degrees in G.

Following is a version of Lemma 2.2 for non-regular hypergraphs.

Lemma 3.1 *Let G be a t-uniform linear hypergraph with r vertices. Then for all $H \in \mathcal{S}_i(G)$*

$$\max_{v \in V_H} \delta_H(v) \geq \max\left\{ \left\lceil \sigma(i)\Big(\frac{ti}{r}\,\frac{\sigma(i)}{\sigma(r)} - 1\Big) \right\rceil, 0 \right\}, \tag{18}$$

where $\sigma(i) = \sigma_G(i)$.

Proof. Replace $\sum_{e \in E_G} \beta_e = si$ with $\sum_{e \in E_G} \beta_e \geq i\sigma(i)$ and $|E_G| = rs/t$ with $|E_G| = r\sigma(r)/t$ in the proof of Lemma 2.2. □

For the case of s-regular hypergraphs $\sigma_G(i) = s$, so Lemma 2.2 is a special case of Lemma 3.1.

Next we combine Lemma 3.1 with Lemma 2.1, to obtain the following non-regular counterpart of Theorem 1.1.

Theorem 3.2 *Let G be a t-uniform linear hypergraph with r vertices. The number of independent sets $I(G)$ in G satisfies*

$$\begin{aligned} \log_2 I(G) &\leq i_0 + \frac{r}{t} \cdot O\Big(\frac{\log^2(ts)}{s_1^2/s}\Big) && (19)\\ &\leq \frac{r}{t} \cdot \frac{s}{s_0} \cdot \Big(1 + O\Big(\frac{\log^2(ts)}{s_1}\Big)\Big) && (20)\\ &\leq \frac{r}{t} \cdot \frac{s}{s_0} \cdot \Big(1 + O\Big(\frac{t\log^2(ts)}{s}\Big)\Big), && (21) \end{aligned}$$

where $s = \sigma_G(r)$ is the average degree in G, i_0 is the largest i for which $i\sigma_G(i) \le rs/t$, $s_0 = \sigma_G(i_0)$, and $s_1 = \sigma_G(i_0+1)$.

Proof. We proceed as in the proof of Theorem 1.1, but this time we let $\Delta(i)$ equal the right-hand side of (18). Also, let $0 = m_0 < m_1 < \ldots < m_\ell = s(t-1)$ be the values taken on by $\Delta(i)$ as i ranges from 0 to r.

Denote by i_j the largest i for which $\Delta(i) = m_j$; in particular, for $j = 0$ we get that i_0 is indeed the largest i for which $i\sigma_G(i) \le rs/t$, and for $j = \ell$ we get $i_\ell = r$. We note that $\sigma(i) = \sigma_G(i)$ is non-decreasing in i and hence so is $\sigma(i)(ti\sigma(i)/(rs) - 1)$. Therefore, i_j is the largest integer i satisfying

$$\sigma(i)\left(\frac{ti}{r}\frac{\sigma(i)}{s} - 1\right) \le m_j$$

or, equivalently, the largest integer i satisfying

$$i \le \left(\frac{m_j}{\sigma(i)} + 1\right)\frac{rs}{t\sigma(i)} \,. \tag{22}$$

This characterization of i_j implies that

$$i_j > \left(\frac{m_j}{\sigma(i_j+1)} + 1\right)\frac{rs}{t\sigma(i_j+1)} - 1. \tag{23}$$

By (22) and (23) we have, for $j \ge 1$,

$$\begin{aligned} i_j - i_{j-1} &\le \frac{rs}{t}\left(\frac{m_j}{(\sigma(i_j))^2} - \frac{m_{j-1}}{(\sigma(i_{j-1}+1))^2} + \frac{1}{\sigma(i_j)} - \frac{1}{\sigma(i_{j-1}+1)}\right) + 1 \\ &\le \frac{rs}{t(\sigma(i_j))^2}(m_j - m_{j-1}) + 1 & (24) \\ &\le \frac{rs}{t(\sigma(i_0+1))^2}(m_j - m_{j-1}) + 1 & (25) \\ &= \frac{rs}{ts_1^2}(m_j - m_{j-1}) + 1, & (26) \end{aligned}$$

where (24) and (25) follow from the fact that $\sigma(i)$ is non-decreasing in i and that $i_0 + 1 \le i_{j-1} + 1 \le i_j$.

Inequality (13) from the proof of Theorem 1.1 applies verbatim here, and incorporating the bound (26) on $i_j - i_{j-1}$ yields

$$\begin{aligned} \log_2 f_r(G) &\le i_0 + \sum_{j=1}^{\ell}\left((m_j - m_{j-1})\frac{rs}{ts_1^2} + 1\right)\log_2 \alpha_{m_j} \\ &\le i_0 + \frac{r}{t}\cdot O\left(\frac{\log^2(ts)}{s_1^2/s}\right), & (27) \end{aligned}$$

where (27) follows from the same reasoning used to obtain (17): the only difference is that here $r \geq m_\ell = (t-1)s \geq (t-1)s_1^2/s$, which we need to assert that $rs/(ts_1^2)$ is bounded away from 0.

Turning to (20), by the definition of i_0 we get that $i_0 s_0 = i_0 \sigma(i_0) \leq rs/t$, i.e., $i_0 \leq (r/t)(s/s_0)$. In addition, since $\sigma(i)$ is non-decreasing in i we have $s_0 s_1 \leq s_1^2$. Combining these two observations with (19) yields (20). Finally, the definition of i_0 also implies that $rs_1 \geq (i_0+1)s_1 > rs/t$; so, $s_1 > s/t$, which readily leads to (21). □

In general, if more is known about the behavior of $\sigma_G(i)$ for $i > i_0$, the $O(\cdot)$ term in (19) can be improved. We obtained (19) by using the pessimistic bound of $\sigma_G(i) \geq \sigma_G(i_0+1)$ for $i > i_0$. We do note, however, that (19) is tight to first order (the i_0 term) for a bipartite graph G in which the degree of any 'left' vertex is smaller than the degree of any 'right' vertex. In such a graph, there are necessarily more left vertices than right vertices and i_0 is easily seen to be the number of left vertices, which in turn is smaller than $\log_2 I(G)$.

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