Lower Bounds on the Anticipation of Encoders for Input-Constrained Channels*

Gitit Ruckenstein Ron M. Roth

Computer Science Department
Technion, Haifa 32000, ISRAEL
e-mail: { gitit, ronny }@cs.technion.ac.il.

Abstract

An input-constrained channel S is defined as the set of words generated by a finite labeled directed graph. It is shown that every finite-state encoder with finite anticipation (i.e., with finite decoding delay) for S can be obtained through state-splitting rounds applied to *some* deterministic graph presentation of S, followed by a reduction of equivalent states. Furthermore, each splitting round can be restricted to follow a certain prescribed structure. This result, in turn, provides a necessary and sufficient condition on the existence of finite-state encoders for S with a given rate $p : q$ and a given anticipation a.

A second condition is derived on the existence of such encoders; this condition is only necessary, but it applies to *every* deterministic graph presentation of S. Based on these two conditions, lower bounds are derived on the anticipation of finite-state encoders. Those lower bounds improve on previously known bounds and, in particular, they are shown to be tight for the common rates used for the $(1,7)$-RLL and $(2,7)$-RLL constraints.

Keywords: Anticipation; Constrained systems; Decoding delay; Decoding look-ahead; Finite-state encoders; Input-constrained channels; Run-length-limited (RLL) constraints; State-splitting algorithm.

1 Introduction

Input-constrained channels are models for describing the read-write requirements of secondary storage systems, such as magnetic disks or optical devices. A widely-used family of

*This work was presented in part at the *IEEE International Symposium on Information Theory*, Ulm, Germany, 1997, and was supported in part by grant No. 95-522 from the United-States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

constraints is the family of (d, k)-*run-length-limited (RLL) constraints*, where each run of 0's between two consecutive 1's in a binary sequence must have length at most k and at least d (the lower bound d does not apply to the first and last runs in the sequence). The $(1, 7)$-RLL, $(2, 7)$-RLL, and $(2, 10)$-RLL constraints are examples of constraints that can be found in commercial systems: the first two constraints appear in magnetic applications, whereas the latter appears in the compact disk [11, Chapter 2] and DVD [12]. A related family of constraints is the class of *multiple-spaced RLL constraints*, which are characterized by three parameters (d, k, s). A binary sequence satisfies such a constraint if the length of every run of 0's between two consecutive 1's is of the form $d + is$, where i is a nonnegative integer such that $k \geq d + is$. Codes with $s = 2$ were shown to have some practical value in magnetic and magneto-optical recording [17].

In all the constraints we will be interested in, the set of allowed sequences is obtained by reading the labels of paths in a labeled directed graph which presents the given constraint. This set of sequences is often referred to as a *constrained system* or simply a *constraint*.

A *rate* $p : q$ *finite-state encoder for a constraint* S accepts input blocks of p bits and generates q-blocks of output symbols, where the output q-block produced at time slot t and the state of the encoder at time slot $t{+}1$ (next state) are both determined by the encoder state at time slot t (current state) and by the input p-block at time slot t. The sequence obtained by concatenating the output blocks of the encoder belongs to the constraint S. The ratio p/q is bounded from above by the *capacity* of the constraint [18].

The encoders of interest are *lossless*, namely, starting in any encoder state, any two distinct sequences of input p-blocks that lead the encoder to the same terminal state are encoded into distinct output sequences; this property allows the reconstruction of the input sequence provided that the initial state, output sequence, and terminal state are known to the decoder. A stronger requirement (which is essential in practical applications) is that the encoders have *finite anticipation*: the anticipation of an encoder is the smallest integer $\mathcal{A}$ such that every two sequences of $\mathcal{A}+1$ input p-blocks that are encoded from the same initial state into the same output sequence have the same first input p-block. Such an encoder can be decoded without the knowledge of the terminal state: by looking ahead at $\mathcal{A}$ upcoming output q-blocks, the decoder can reconstruct the current input p-block and the next state, and iteration of this process recovers the whole sequence of input p-blocks. The anticipation of a given encoder thus measures the smallest look-ahead or delay of any decoder for that encoder.

A special class of encoders with finite anticipation is that of (m, a)-*sliding-block decodable encoders*: the input p-block at time slot t is reconstructed by applying a decoding function (which is independent of t) to the received output q-blocks at time slots $t{-}m, t{-}m{+}1, \ldots, t{+}a{-}1, t{+}a$, for some prescribed m and a. With the exception of certain degenerate cases, the anticipation of such an encoder must be at least a; therefore, any lower bound on the anticipation implies a lower bound on the parameter a. (We point out that even though the parameter a is always nonnegative, there are examples where the parameter m can be negative; see [15, p. 1688].) Sliding-block decodable encoders are desirable due to their simple decoder structure and the limited error propagation of the decoding process.

One of the most notable schemes known for constructing finite-state encoders is the *state-splitting algorithm* due to Adler, Coppersmith, and Hassner [1]. This algorithm allows to construct encoders at any rational rate up to the capacity of the constraint. Moreover, the encoders obtained by the algorithm have finite anticipation, and in many cases they are sliding-block decodable.

When using the state-splitting algorithm, there are often many choices that the designer can make during the course of the algorithm, and it is not clear which choice yields an encoder with (say) the smallest anticipation or the smallest number of states. A graph presentation of the constraint serves as an input to the state-splitting algorithm. Since the graph presentation of a constraint is not unique, the designer needs also to decide which presentation is preferable to start with.

In this work, we first establish the universality of the state-splitting algorithm for finite-state encoders: we prove that every finite-state encoder with finite anticipation can be obtained by the state-splitting algorithm, followed by an operation of *reduction of states*. In a way, this result is an extension of the work of Ashley and Marcus in [2], where they showed the universality of the state-splitting algorithm to the special case of sliding-block decodable encoders.

Then, by further characterizing the state-splitting rounds to be applied throughout the course of the algorithm, we derive lower bounds on the anticipation of any rate $p : q$ finite-state encoder for a given constraint S. Lower bounds on the anticipation were obtained previously in [14] and [4] (see Theorems 2.4 and 2.5 below). Our new bounds improve on those previous results. In particular, our bounds are shown to be tight in certain practical cases: we show that any rate $2 : 3$ finite-state encoder for the $(1, 7)$-RLL constraint must have anticipation at least 2 and that any rate $1 : 2$ encoder for the $(2, 7)$-RLL constraint, as well as any rate $2 : 5$ encoder for the $(2, 18, 2)$-RLL constraint, must have anticipation at least 3. All the three bounds are attainable by known constructions.

This paper is organized as follows. Section 2 summarizes the necessary background. Our main results are Theorem 3.1 and Theorem 3.2, which are stated in Section 3. In Section 4 we treat the special case where the rate of the encoder attains the capacity of the constraint. The results obtained for this case are then used in Sections 5 to prove our main theorems. Section 6 contains examples that demonstrate how our results can be applied to obtain lower bounds on the anticipation of any encoder for a given constraint at a given rate. Section 7 is a conclusion section.

2 Background

Sections 2.1–2.3 provide a summary of terms and known results on finite-state encoders for constraints. Full details and formal proofs can be found in [15]. In Section 2.4, we introduce some additional terms and properties that will be later used in this paper.

2.1 Graph presentations

A *finite labeled directed graph* (or simply a *graph*) $G = (V, E, L)$ consists of a nonempty finite set of states $V = V(G)$, a finite set of directed edges $E = E(G) \subseteq V \times V$ (where parallel edges are allowed), and (output) labeling $L : E \to \Sigma$ on the edges taken from an alphabet Σ. The initial state and the terminal state of an edge e will be denoted by $\sigma_G(e)$ and $\tau_G(e)$, respectively.

A *path* in G is a finite sequence of edges $e_1 e_2 \ldots e_\ell$ such that $\sigma_G(e_i) = \tau_G(e_{i-1})$. A single state in G is defined as a path of zero length. For a nonempty path $\pi = e_1 e_2 \ldots e_\ell$ we denote by π^* the truncated path $e_2 e_3 \ldots e_\ell$. A *word* over Σ is a finite sequence of symbols from Σ. A word $\boldsymbol{w}$ is said to be generated by a path π in G if $\boldsymbol{w}$ is obtained by reading the labels of the edges in π. The empty word, denoted $\boldsymbol{\epsilon}$, is defined as the (unique) word of zero length, and for a nonempty word $\boldsymbol{w} = w_1 w_2 \ldots w_\ell$ we denote by $\boldsymbol{w}^*$ the truncated word $w_2 w_3 \ldots w_\ell$.

Given a graph G, the *constraint* $S(G)$ is the set of words generated by paths in G, and G is called a graph presentation of $S(G)$. For example, Figure 1 shows a graph presentation of the $(1, 7)$-RLL constraint.

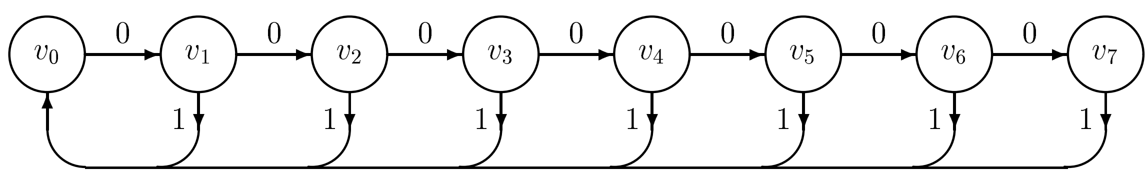

Figure 1: Graph presentation of the (1,7)-RLL constraint.

An *adjacency matrix* of a graph G, denoted A_G, is a square matrix whose rows and columns are indexed by $V(G)$ and $(A_G)_{u,v}$ is the number of edges $e \in E(G)$ with $\sigma_G(e) = u$ and $\tau_G(e) = v$. By the Perron-Frobenius Theorem, A_G has a nonnegative real eigenvalue, denoted $\lambda(A_G)$, that is at least as large as the absolute value of any other eigenvalue of A_G. For a regular graph G with out-degree n (i.e., a graph in which every state has exactly n outgoing edges) we have $\lambda(A_G) = n$.

A graph G is called *deterministic* if the labels of the outgoing edges of each state in G are distinct. Every constraint has a deterministic graph presentation. A graph is *lossless* if for every (ordered) pair (u, v) of states in $V(G)$, any word can be generated in G by at most one path that starts in u and terminates in v. A graph G has a *finite anticipation* if there is an integer N such that any two paths of length $N+1$ in G that have the same initial state and generate the same word must have the same initial edge. The smallest such N, if any, is called the anticipation of G and denoted $\mathcal{A}(G)$. If no such N exists we define $\mathcal{A}(G) = \infty$. Finite anticipation implies losslessness if every state in G has at least one outgoing edge.

A graph G is *irreducible* if for every (ordered) pair of states (u, v) in G there is a path from u to v in G. An *irreducible component* of a graph G is a maximal irreducible subgraph of G (namely, an irreducible component is not a proper subgraph of any other irreducible subgraph of G). An *irreducible sink* of G is an irreducible component H from which there are no outgoing edges to any other irreducible component of G. Every graph consists of finitely-many irreducible components, at least one of which is an irreducible sink.

A constraint S is *irreducible* if for every two words $\boldsymbol{w}, \boldsymbol{w}' \in S$ there is a word $\boldsymbol{z} \in S$ such that $\boldsymbol{w}\boldsymbol{z}\boldsymbol{w}' \in S$. It is known that a constraint is irreducible if and only if it can be presented by an irreducible graph.

Given a graph G and a state $u \in V(G)$, the *follower set* of u, denoted $\mathcal{F}_G(u)$, is the set of words generated by paths in G that start in u. Two states u and u' in graphs G and G', respectively, are *equivalent* if $\mathcal{F}_G(u) = \mathcal{F}_{G'}(u')$, and state u is *dominated* by u', denoted $u \preceq u'$, if $\mathcal{F}_G(u) \subseteq \mathcal{F}_{G'}(u')$.

By a *reduction* of equivalent states in a graph G we mean replacing a set of equivalent states with one representative of this set: the edges incoming to the replaced states are redirected into that representative state. Reducing equivalent states in a graph G preserves the constraint $S(G)$. A *reduced graph* is a graph that does not contain distinct equivalent states. Given a deterministic graph G, there is an algorithm, known as the *Moore algorithm*, for constructing a reduced deterministic graph H where $S(H) = S(G)$ (see [15, Section 2.6.2]).

Let S be a constraint. The *Shannon cover* of S, denoted G_S, is a deterministic presentation of S with a smallest number of states. The following proposition summarizes well-known properties of the Shannon cover of irreducible constraints.

Proposition 2.1 *Let S be an irreducible constraint.*

(i) The Shannon cover G_S is the unique graph presentation (up to a graph isomorphism) of S that is deterministic, irreducible, and reduced.

(ii) The set of follower sets of the states of any irreducible deterministic presentation of S is the same as that of G_S.

(iii) Let G' present a constraint S' that contains S. Then for every state $u \in V(G_S)$ there is at least one state $u' \in V(G')$ such that $\mathcal{F}_{G_S}(u) \subseteq \mathcal{F}_{G'}(u')$.

The kth *power graph* of a graph G, denoted G^k, is a graph with $V(G^k) = V(G)$ and an edge for each path of length k in G, labeled by the word generated by that path. The constraint presented by G^k is denoted by S^k, and $A_{G^k} = A_G^k$.

For a constraint S, denote by $N(\ell; S)$ the number of words in S of length ℓ. The *(Shannon) capacity* of S, denoted $\mathsf{cap}(S)$, is defined by

$$\mathsf{cap}(S) = \lim_{\ell \to \infty} \frac{1}{\ell} \log_2 N(\ell; S) .$$

The limit indeed exists. If G is any lossless presentation of S, then

$$\mathsf{cap}(S) = \log_2 \lambda(A_G) .$$

Let S be a constraint and let n be a positive integer. An (S, n)-*encoder* is a graph $\mathcal{E}$ such that (i) $S(\mathcal{E}) \subseteq S$, (ii) $\mathcal{E}$ is lossless, and (iii) $\mathcal{E}$ is regular of out-degree n. (S, n)-encoders exist if and only if $\log_2 n \le \mathsf{cap}(S)$: necessity follows from Shannon's converse-to-coding theorem and sufficiency follows from the state-splitting algorithm which we review in Section 2.2. A *tagged* (S, n)-encoder is an (S, n)-encoder where each edge is assigned a tag ("input label") from an alphabet of size n such that the outgoing edges from each state have distinct tags.

Recalling the definition of finite-state encoders in Section 1, a rate $p : q$ finite-state encoder for a constraint S is a tagged $(S^q, 2^p)$-encoder, where the tags are drawn from $\{0, 1\}^p$. Figure 2 shows a rate $1 : 2$ six-state encoder for the $(2, 7)$-RLL constraint, where the notation $s/\boldsymbol{w}$ next to each edge specifies the input tag s and the label $\boldsymbol{w}$. The encoder in the figure is due to Franaszek [8], [10] and has anticipation 3.

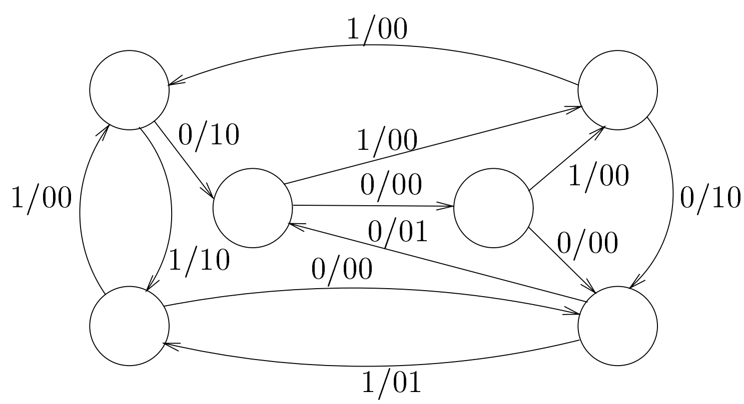

Figure 2: Rate $1 : 2$ six-state encoder for the $(2, 7)$-RLL constraint.

As pointed out in Section 1, the anticipation of a tagged (S, n)-encoder $\mathcal{E}$ is the amount of look-ahead required by any decoder for $\mathcal{E}$. A *state-dependent* decoder restores the input tag sequence by reconstructing the path of $\mathcal{E}$ by which the output sequence was generated. The (m, a)-sliding-block decodable encoders mentioned in Section 1 are a special case of encoders that have *state-independent* decoders. If $\mathcal{E}$ is an irreducible (m, a)-sliding-block decodable encoder then $a \ge \mathcal{A}(\mathcal{E})$.

Let S be a constraint that is presented by a graph G. It follows from [14, Proposition 3] that any (S, n)-encoder contains an irreducible sink that is an (S', n)-encoder, where S'

is presented by an irreducible component of G. Hence, there is no loss of generality in limiting our study—in particular obtaining lower bounds on the anticipation of encoders—to irreducible constraints.

2.2 State splitting

The *state-splitting algorithm* due to Adler, Coppersmith, and Hassner [1] is a well-known method for constructing (S,n)-encoders with finite anticipation whenever $\log_2 n \le \mathsf{cap}(S)$. As demonstrated in [14], [2], and [4], it has also been proven to be useful as an analysis tool for such encoders. The algorithm will play a similar role in this paper for obtaining new lower bounds on the anticipation.

The state-splitting algorithm turns a deterministic presentation G of S into a new graph in which every state has at least n outgoing edges. For the sake of completeness, we summarize the algorithm below, following the next definitions.

Let G be a graph. A *round of (state out-)splitting* of G produces a new graph G' from G through a partition of the set E_u of outgoing edges of each state u in G into $N(u) \ge 1$ disjoint nonempty sets $E_u = E_u^{(1)} \cup E_u^{(2)} \cup \ldots \cup E_u^{(N(u))}$. In the new graph G', the states are the descendant states $u^{(1)}, u^{(2)}, \ldots, u^{(N(u))}$ of every state u of G, and for every edge $u \stackrel{b}{\to} v$ of G that belongs to $E_u^{(i)}$, the graph G' is endowed with the edges

$$u^{(i)} \stackrel{b}{\to} v^{(r)} , \quad r = 1, 2, \ldots, N(v) .$$

Proposition 2.2 *Let G' be a graph obtained from G by a round of splitting. Then,*

(i) $S(G') = S(G)$.

(ii) $\mathcal{A}(G') \le \mathcal{A}(G) + 1$.

(iii) If G is an irreducible graph, then so is G'.

The following notion is used in the state-splitting algorithm as a guide for how to split the states. Given a nonnegative integer square matrix A and an integer n, an *(A,n)-approximate eigenvector* is a nonnegative integer vector $\boldsymbol{x}$ such that $\boldsymbol{x} \ne \mathbf{0}$ and $A\boldsymbol{x} \ge n\boldsymbol{x}$, where the inequality holds component by component. By the Perron-Frobenius theorem it follows that there exist (A,n)-approximate eigenvectors if and only if $n \le \lambda(A)$. It is easy to see that every (A,n)-approximate eigenvector is also an (A^k, n^k)-approximate eigenvector for every positive integer k.

An *(A,n)-integral eigenvector* is an (A,n)-approximate eigenvector $\boldsymbol{x}$ that satisfies $A\boldsymbol{x} = n\boldsymbol{x}$; i.e., $\boldsymbol{x}$ is a nonnegative integer eigenvector of A associated with the eigenvalue n.

For a nonnegative vector $\boldsymbol{y}$, denote by $\|\boldsymbol{y}\| = \|\boldsymbol{y}\|_\infty$ the largest component of $\boldsymbol{y}$. Given a nonnegative integer square matrix A and positive integers n and B, there is an algorithm, known as Franaszek's algorithm, for finding an (A, n)-approximate eigenvector $\boldsymbol{x}$ such that $\|\boldsymbol{x}\| \le B$, whenever one exists (and Franaszek's algorithm returns a vector $\boldsymbol{x}$ that satisfies $\boldsymbol{x}' \le \boldsymbol{x}$ for any other (A, n)-approximate eigenvector $\boldsymbol{x}'$; see [15, p. 1671]).

Let A_G be the adjacency matrix of a graph G and let $\boldsymbol{x} = [x_u]_{u \in V(G)}$ be an (A_G, n)-approximate eigenvector. The value x_u is referred to as the *weight* of state u in G. An $\boldsymbol{x}$*-consistent round of splitting* of G is a round of splitting where each descendant state, $u^{(i)}$, can be assigned a nonnegative integer weight $x_u^{(i)}$ such that the following holds: (i) $x_u^{(1)} + x_u^{(2)} + \ldots + x_u^{(N(u))} = x_u$ for every $u \in V(G)$, and (ii) if G' is the resulting split graph, then the vector $\boldsymbol{x}'$ defined by $(\boldsymbol{x}')_{u^{(i)}} = x_u^{(i)}$ is an $(A_{G'}, n)$-approximate eigenvector. We say that $\boldsymbol{x}'$ is the approximate eigenvector that is *induced* by the splitting.

The state-splitting algorithm is summarized in Figure 3. It is shown in [1] that there always exists an $\boldsymbol{x}$-consistent round of splitting that can be applied in Step 3a. If the graph H in Step 3 is such that the all-one vector is an (A_H, n)-approximate eigenvector, then each state in H has out-degree at least n. We say in this case that H is obtained by an n*-full splitting* of G consistently with the (A_G, n)-approximate eigenvector $[x_u]_{u \in V(G)}$. Such a graph H has a regular subgraph $\mathcal{E}$ with out-degree n. By Proposition 2.2 it follows that $S(\mathcal{E}) \subseteq S$ and the anticipation of $\mathcal{E}$ is at most the final value of a.

---

1. Given a constraint S and a positive integer n such that $\log_2 n \le \mathsf{cap}(S)$, select an irreducible deterministic graph presentation G of S and an (A_G, n)-approximate eigenvector $[x_u]_{u \in V(G)}$.
2. Delete from G all the states of weight zero, including their incident edges, and let H be an irreducible sink of the resulting graph. Let $\boldsymbol{x}$ be the (A_H, n)-approximate eigenvector $[x_u]_{u \in V(H)}$.
3. Set $\mathsf{a} \leftarrow 0$, and repeat the following as long as the all-one vector is *not* an (A_H, n)-approximate eigenvector:
   (a) Apply an $\boldsymbol{x}$-consistent round of splitting to H to obtain a new irreducible graph H' and an induced $(A_{H'}, n)$-approximate eigenvector $\boldsymbol{x}'$.
   (b) $H \longleftarrow H'$; $\boldsymbol{x} \longleftarrow \boldsymbol{x}'$; $\mathsf{a} \leftarrow \mathsf{a} + 1$.
4. Remove excess edges from H to get an (S, n)-encoder $\mathcal{E}$.

Figure 3: State-splitting algorithm.

The state-splitting algorithm can produce different (S, n)-encoders, depending on the selection of the initial deterministic presentation G and approximate eigenvector $[x_u]_{u \in V(G)}$, and the particular rounds of splitting performed throughout the course of the algorithm.

The following lemma describes the connection between the state-splitting process of a graph and the state-splittings of its power graphs. The proof of the lemma is similar to that of [13, Theorem 3.4].

Lemma 2.3 *Suppose that an irreducible graph G can be split in k rounds consistently with an (A_G, n)-approximate eigenvector $\boldsymbol{x}$, yielding a graph H and an induced approximate* *eigenvector $\boldsymbol{y}$. Then the graph G^k can be split in one round consistently with the vector $\boldsymbol{x}$, yielding the graph H^k and the induced approximate eigenvector $\boldsymbol{y}$. Moreover, for every state $u \in G$, the number of descendant states of u in H is equal to the number of descendant states of $u \in G^k$ in H^k.*

2.3 Previous bounds on the anticipation of encoders

We summarize below the known lower bounds on the anticipation of any (S, n)-encoder. The following lower bound on the anticipation was obtained in [14].

Theorem 2.4 *Let S be a constraint presented by a deterministic graph G and let $\mathcal{E}$ be an (S, n)-encoder. The anticipation of any (S, n)-encoder is at least* $\min_{\boldsymbol{x}} \log_n \|\boldsymbol{x}\|$ *where the minimum is taken over all (A_G, n)-approximate eigenvectors $\boldsymbol{x}$.*

Equivalently, the existence of an (S, n)-encoder with anticipation a implies the existence of some (A_G, n)-approximate eigenvector $\boldsymbol{x}$ such that $\|\boldsymbol{x}\| \leq n^{\mathsf{a}}$. This necessary condition holds for every deterministic presentation G of S, in particular the Shannon cover of S.

The lower bound of Theorem 2.4 can be computed easily using Franaszek's algorithm for computing approximate eigenvectors.

Another lower bound on the anticipation of (S, n)-encoders is implied by the following result, taken from [4].

Theorem 2.5 *Let S, G, and n be as in Theorem 2.4. The anticipation of any (S, n)-encoder is at least the smallest integer* a *for which there exists an (A_G, n)-approximate eigenvector $\boldsymbol{x}$ (which is also an $(A_G^{\mathsf{a}}, n^{\mathsf{a}})$-approximate eigenvector) such that G^{a} can be n^{a}-fully split in one round consistently with $\boldsymbol{x}$.*

Conversely, it is shown in [4] that for the smallest integer a in Theorem 2.5, there is an (S, n)-encoder with anticipation not greater than $2\mathsf{a} - 1$. Other upper bounds on the anticipation, which are linear in the number of states $V(G)$, are derived in [5], [6], [3], and [7]. In contrast, the known general upper bound on the anticipation of (S, n)-encoders constructed by the state-splitting algorithm can be exponential in $V(G)$; see [15, Section 6.2] for more details.

Our main results (Theorem 3.1 and Theorem 3.2 below) are improvements to Theorems 2.4 and 2.5 and include those previous results as special cases.

2.4 Strong equivalence and graph expansion

We define two states u and u' as 0-*strongly equivalent* if they are equivalent states. States u and u' are t-*strongly equivalent* if the following conditions hold:

1. A one-to-one and onto mapping $\varphi : E_u \to E_{u'}$ can be defined from the set of outgoing edges of u to the set of outgoing edges of u', such that for every $e \in E_u$, both e and $\varphi(e)$ have the same label.

2. For every $e \in E_u$, the terminal states of e and $\varphi(e)$ are $(t-1)$-strongly equivalent.

States that are t-strongly equivalent are also r-strongly equivalent (and in particular they are equivalent states) for every $r < t$.

We say that two states are *strongly equivalent states* if for every $t \geq 0$ the states are t-strongly equivalent. So, when states are strongly equivalent, the infinite trees of paths that start in those states are the same. In a deterministic graph, two states are equivalent if and only if they are strongly equivalent. On the other hand, in a nondeterministic graph there may be two states that are equivalent but not strongly equivalent. See, for example, the states u and v in Figure 4.

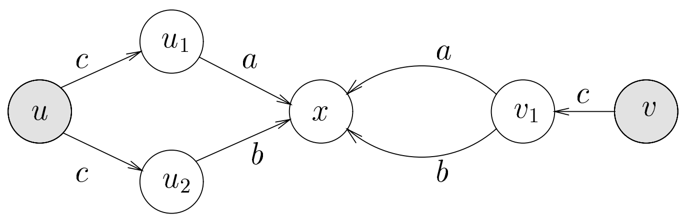

Figure 4: Equivalent states that are not strongly equivalent.

The following proposition states that the anticipation of a graph cannot increase as a result of reducing strongly equivalent states.

Proposition 2.6 *Let H' be an irreducible graph and let H be a graph that can be obtained from H' by a reduction of strongly equivalent states. Then $\mathcal{A}(H) \leq \mathcal{A}(H')$.*

Proof. The case where $\mathcal{A}(H') = \infty$ is trivial. Assume that $\mathcal{A}(H') < \infty$ and suppose to the contrary that $\mathcal{A}(H) > \mathcal{A}(H')$. Then there must exist some state u in H and two distinct paths of length $\mathcal{A}(H') + 1$ that start in u with distinct edges and generate the same word $\boldsymbol{w}$. Let e_1 and e_2 be the first edges of these two paths, respectively, and let $v_1 = \tau_H(e_1)$ and $v_2 = \tau_H(e_2)$. Under our assumption, e_1 and e_2 have the same label, say b (b is the first symbol in $\boldsymbol{w}$).

Let u' be a state of H' that is reduced to state u in H. The states u' and u are strongly equivalent states and, as a result, there are two distinct edges labeled b outgoing from u' to two different states, v'_1 and v'_2. States v'_1 and v'_2 are strongly equivalent to v_1 and v_2, respectively. Since v_1 is equivalent to v'_1, there must be a path of length $\mathcal{A}(H')$ starting in v'_1 that generates the word $\boldsymbol{w}^*$ (i.e., the word obtained by deleting the first symbol, b, from

$\boldsymbol{w}$). The same argument holds for v_2'. Hence, there are two paths of length $\mathcal{A}(H')+1$ in H' that start in u' and generate $\boldsymbol{w}$, yet those paths do not start with the same edge. This contradicts the definition of $\mathcal{A}(H')$. We therefore conclude that $\mathcal{A}(H) \le \mathcal{A}(H')$. ∎

Let G be a graph. An *expansion* of G is a graph G' obtained from G as follows: for each state u of G we define $K(u) \ge 1$ *duplicate* states $u_1, u_2, \ldots, u_{K(u)}$ in G', and for every edge $u \xrightarrow{b} v$ of G we endow G' with the edges

$$u_r \xrightarrow{b} v_{i_r} \ , \quad r = 1, 2, \ldots, K(u) \ ,$$

where each i_r can be any element in $\{1, 2, \ldots, K(v)\}$. (We mention that a round of state *in*-splitting [15, p. 1694] results in a special case of expansion where all the terminal states v_{i_r} are the same.)

All the duplicates of a state in the expansion graph are strongly equivalent. Conversely, if a graph G is obtained from a graph G' by reducing *all* the strongly equivalent states in G', then G' is an expansion of G. From Proposition 2.1(ii) we thus have the following.

Proposition 2.7 *Every irreducible deterministic presentation of a constraint S is an expansion of the Shannon cover G_S of S.*

Let G' be an irreducible deterministic graph that is an expansion of an irreducible deterministic graph G. A function $f : V(G') \to V(G)$ can be defined that maps every state of G' to its parent state in G. Given an (A_G, n)-approximate eigenvector $\boldsymbol{x}$, an $(A_{G'}, n)$-approximate eigenvector $\boldsymbol{x}'$, can be defined by

$$(\boldsymbol{x}')_{u'} = (\boldsymbol{x})_{f(u')} \ , \quad u' \in V(G') \ .$$

The vector $\boldsymbol{x}'$ is then referred to as an *expansion* of $\boldsymbol{x}$.

Proposition 2.8 *Let G be an irreducible graph and let H be a graph obtained from G by m rounds of splitting, consistently with an (A_G, n)-approximate eigenvector $\boldsymbol{x}$. Suppose that G' is an irreducible expansion of G. Then G' can be split in m rounds consistently with an expansion $\boldsymbol{x}'$ of $\boldsymbol{x}$ to produce a graph H' that is an expansion of H. The induced $(A_{H'}, n)$-approximate eigenvector $\boldsymbol{y}'$ is an expansion of the induced (A_H, n)-approximate eigenvector $\boldsymbol{y}$.*

Proof. We prove the result for the first round, and the rest will follow inductively. The outgoing edges of a state u_r in G' match the outgoing edges of the parent state u in G in their number, labeling, and the weights of their terminal states according to $\boldsymbol{x}$ and $\boldsymbol{x}'$, respectively. Therefore, when splitting a state u_r in G', we can 'copy' the splitting of state u in G. That is, if u is split in the first round into $u^{(1)}, u^{(2)}, \ldots, u^{(N(u))}$, with respective weights $(\boldsymbol{y})_{u^{(1)}}, (\boldsymbol{y})_{u^{(2)}}, \ldots, (\boldsymbol{y})_{u^{(N(u))}}$, then u_r can be split into $u_r^{(1)}, u_r^{(2)}, \ldots, u_r^{(N(u))}$ with respective weights $(\boldsymbol{y}')_{u_r^{(s)}} = (\boldsymbol{y})_{u^{(s)}}$. Every descendant state $u_r^{(s)}$ of u_r inherits the outgoing edges that correspond to those assigned to the descendent state $u^{(s)}$ of u. ∎

3 Statement of main results

Our first main result is as follows.

Theorem 3.1 *Let S be an irreducible constraint and let n be a positive integer where $\mathsf{cap}(S) \geq \log_2 n$. Suppose there exists some irreducible (S, n)-encoder $\mathcal{E}$ with $\mathcal{A}(\mathcal{E}) = \mathsf{a} < \infty$. Then there exists an irreducible deterministic (not necessarily reduced) presentation $\mathcal{G}$ of S and an $(A_{\mathcal{G}}, n)$-approximate eigenvector $\boldsymbol{x}$ that satisfy the following:*

(i) $\|\boldsymbol{x}\| \leq n^{\mathsf{a}}$.

(ii) The graph $\mathcal{G}$ can be n-fully split consistently with $\boldsymbol{x}$ in a rounds of splitting. After deleting excess edges, an (S, n)-encoder $\mathcal{E}_{\mathcal{G}}$ is obtained with $\mathcal{A}(\mathcal{E}_{\mathcal{G}}) = \mathsf{a}$.

(iii) In each of the splitting rounds, every state is split into at most n states.

(iv) In the ith round, the induced approximate eigenvector $\boldsymbol{x}^{(i)}$ satisfies $\|\boldsymbol{x}^{(i)}\| \leq n^{\mathsf{a}-i}$.

(v) The encoder $\mathcal{E}$ can be obtained from $\mathcal{E}_{\mathcal{G}}$ by a reduction of strongly equivalent states.

The significance of Theorem 3.1 is twofold:

1. Given S and n, the theorem implies a lower bound on the anticipation of any (S, n)-encoder: the anticipation of such encoder is at least the smallest nonnegative integer a for which there exists a presentation $\mathcal{G}$ of S and an $(A_{\mathcal{G}}, n)$-approximate eigenvector $\boldsymbol{x}$ that satisfy conditions (i)–(v) of Theorem 3.1. Specific examples for the computation of such a lower bound are given in Section 6.

2. Theorem 3.1 establishes the universality of the state-splitting algorithm for encoders with finite anticipation: every (S, n)-encoder with finite anticipation can be constructed using the state-splitting algorithm, combined with reductions of (strongly) equivalent states, where the input to the process is some irreducible deterministic presentation $\mathcal{G}$ of S and an $(A_{\mathcal{G}}, n)$-approximate eigenvector $\boldsymbol{x}$.

   A parallel result was derived in [2] for the special case of sliding-block decodable encoders. Given a sliding-block *decoding function* $\mathcal{D}$, it was shown that an (S, n) encoder corresponding to $\mathcal{D}$ can be obtained from some deterministic presentation of S and a respective approximate eigenvector. If the deterministic presentation is fixed to be the Shannon cover of S but no restrictions are forced on the choice of the initial approximate eigenvector, the mentioned result of [2] still holds under some minor changes.

   However, as we show in Appendix A (Example A.1), not always can the Shannon cover be taken as the graph $\mathcal{G}$ in Theorem 3.1. (Note that Theorem 3.1 does force certain restrictions on the initial approximate eigenvector, as well as on the number of splitting rounds and the vectors obtained during the splitting process.)

Our second main result is stated next.

Theorem 3.2 *Let S be an irreducible constraint, let n be a positive integer where $\mathsf{cap}(S) \geq \log_2 n$, and let G be any irreducible deterministic presentation of S. Suppose there exists some irreducible (S, n)-encoder with anticipation $\mathsf{a} < \infty$. Then there exists an (A_G, n)-approximate eigenvector $\boldsymbol{x}$ such that the following holds:*

(i) $\|\boldsymbol{x}\| \leq n^{\mathsf{a}}$.

(ii) For every $k = 1, 2, \ldots, \mathsf{a}$, the states of G^k can be split in one round consistently with the (A_G^k, n^k)-approximate eigenvector $\boldsymbol{x}$, such that the induced approximate eigenvector $\boldsymbol{x}'$ satisfies $\|\boldsymbol{x}'\| \leq n^{\mathsf{a}-k}$, and each of the states in G^k is split into no more than n^k states.

While Theorem 3.1 gives a necessary and sufficient condition on the existence of (S, n)-encoders with a given anticipation a, Theorem 3.2 gives only a *necessary* condition on the existence of such encoders. On the other hand, Theorem 3.2 allows to obtain a lower bound on the anticipation using *any* irreducible deterministic presentation of S—in particular the Shannon cover of S. Therefore, it will typically be easier to compute bounds using Theorem 3.2.

Note that Theorem 2.4 is equivalent to Theorem 3.2(i), while Theorem 2.5 is equivalent to Theorem 3.2(ii) for the special case $k = \mathsf{a}$. In Section 6 we show that Theorem 3.2 (and hence Theorem 3.1) yields stronger bounds than the previously known lower bounds.

4 Encoders attaining capacity

In this section, we deal with the special case of (S, n)-encoders $\mathcal{E}$ where $S = S(\mathcal{E})$ (it can be shown that when S is irreducible, then this is equivalent to having $\log_2 n = \mathsf{cap}(S)$). The results herein will then be used to prove Theorem 3.1 and Theorem 3.2.

Specifically, we prove here the following two propositions.

Proposition 4.1 *Let $\mathcal{E}$ be an irreducible $(S_{\mathcal{E}}, n)$-encoder with anticipation $\mathsf{a} < \infty$, where $S(\mathcal{E}) = S_{\mathcal{E}}$. There exists an irreducible deterministic graph presentation $G_{\mathcal{E}}$ of $S_{\mathcal{E}}$ and an $(A_{G_{\mathcal{E}}}, n)$-integral eigenvector $\boldsymbol{c} = \boldsymbol{c}_{\mathcal{E}}$ such that*

(i) $\boldsymbol{c}$ is strictly positive (i.e., each component of $\boldsymbol{c}$ is strictly positive) and $\|\boldsymbol{c}\| \leq n^{\mathsf{a}}$.

(ii) The graph $G_{\mathcal{E}}$ can be n-fully split consistently with $\boldsymbol{c}$ in a rounds of splitting. The resulting graph, $\mathcal{E}'$, is an $(S_{\mathcal{E}}, n)$-encoder and $\mathcal{A}(\mathcal{E}') = \mathsf{a}$.

(iii) In each of the splitting rounds, every state is split into at most n states.

(iv) *In the ith round of splitting, the induced integral eigenvector* $\boldsymbol{c}^{(i)}$ *satisfies* $\|\boldsymbol{c}^{(i)}\| \le n^{\mathsf{a}-i}$.

(v) *The encoder* $\mathcal{E}$ *can be obtained from* $\mathcal{E}'$ *by a reduction of strongly equivalent states.*

Proposition 4.2 *Let* $\mathcal{E}$, $S_{\mathcal{E}}$, n, a, $G_{\mathcal{E}}$, *and* $\mathbf{c} = \boldsymbol{c}_{\mathcal{E}}$ *be as in Proposition 4.1. For every* $k = 1, 2, \ldots, \mathsf{a}$, *there is a* $\mathbf{c}$*-consistent round of splitting of* $G_{\mathcal{E}}^k$ *with respect to the* $(A_{G_{\mathcal{E}}}^k, n^k)$*-integral eigenvector* $\boldsymbol{c}$, *such that every state of* $G_{\mathcal{E}}^k$ *is split into no more than* n^k *states, and the induced integral eigenvector* $\boldsymbol{c}^{(k)}$ *satisfies* $\|\boldsymbol{c}^{(k)}\| \le n^{\mathsf{a}-k}$.

The rest of this section is devoted to the proofs of these two propositions. The proofs are based on the construction and splitting properties of a so-called *determinizing graph* for a given encoder graph $\mathcal{E}$, namely a deterministic graph $G_{\mathcal{E}}$ such that $S(\mathcal{E}) = S(G_{\mathcal{E}})$. Hereafter, $S_{\mathcal{E}}$, n, $\mathcal{E}$, and a are as in Proposition 4.1.

Let π be a path of length i in $\mathcal{E}$ and $\boldsymbol{w}$ be a word of length ℓ in $S_{\mathcal{E}}$. Denote by $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ the set of all paths $e_1 e_2 \ldots e_{i+\ell}$ in $\mathcal{E}$ such that

$$e_1 e_2 \ldots e_i = \pi \quad \text{and} \quad e_{i+1} e_{i+2} \ldots e_{i+\ell} \text{ generates } \boldsymbol{w} .$$

Observe that when $\ell + i = \mathsf{a} + 1$, the paths in $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ share the same first edge, e_1, even when π has zero length (i.e., π is a state of $\mathcal{E}$).

For $i = 0, 1, \ldots, \mathsf{a}$, define the graph $G_{\mathcal{E}}^{(i)}$ as follows. The states of $G_{\mathcal{E}}^{(i)}$ are all the *nonempty* sets $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ of paths in $\mathcal{E}$ in which π has length i and $\boldsymbol{w}$ has length $\mathsf{a}-i$. The outgoing edges of a state $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ in $G_{\mathcal{E}}^{(i)}$ are given by

$$\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w}) \xrightarrow{b} \mathcal{T}_{\mathcal{E}}((\pi e)^*, (\boldsymbol{w} b)^*) , \tag{1}$$

where b ranges over the symbols for which $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w} b) \ne \emptyset$ and e ranges over the edges in $\mathcal{E}$ labeled by the first symbol of $\boldsymbol{w}$ such that πe is a path in $\mathcal{E}$.

The above definition applies also to $i = \mathsf{a}$, in which case $\boldsymbol{w} = \boldsymbol{\epsilon}$ (the empty word). In the other extreme case of $i = 0$, the paths π are states of $\mathcal{E}$. In this case, we will use the notation $\mathcal{T}_{\mathcal{E}}(u, \boldsymbol{w})$ where $u \in V(\mathcal{E})$. That is, a state $\mathcal{T}_{\mathcal{E}}(u, \boldsymbol{w})$ in $G_{\mathcal{E}}^{(0)}$ is the set of all paths in $\mathcal{E}$ of length a that share their initial state u and generate the same word $\boldsymbol{w}$ of length a.

Let $Z = \mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ be a state in $G_{\mathcal{E}}^{(i)}$. We define the *weight* of Z, denoted $|Z|$, to be the size of Z when regarded as a set of paths of $\mathcal{E}$. We write the weights of the states of $G_{\mathcal{E}}^{(i)}$ as a vector $\boldsymbol{c}^{(i)} = \boldsymbol{c}_{\mathcal{E}}^{(i)} = [(\boldsymbol{c}^{(i)})_Z]_{Z \in V(G^{(i)})}$, where

$$(\boldsymbol{c}^{(i)})_Z = |Z| , \quad Z \in V(G_{\mathcal{E}}^{(i)}) .$$

Figure 5(a) shows a state $Z \in V(G_{\mathcal{E}}^{(0)})$ for an encoder $\mathcal{E}$ with $n = \mathsf{a} = 2$, and Figure 5(b) shows the outgoing edges of Z in $G_{\mathcal{E}}^{(0)}$. In those figures, the paths in the sets $\mathcal{T}_{\mathcal{E}}(u, \boldsymbol{w})$ are merged into trees, and the numbers indicate the weights of the states.

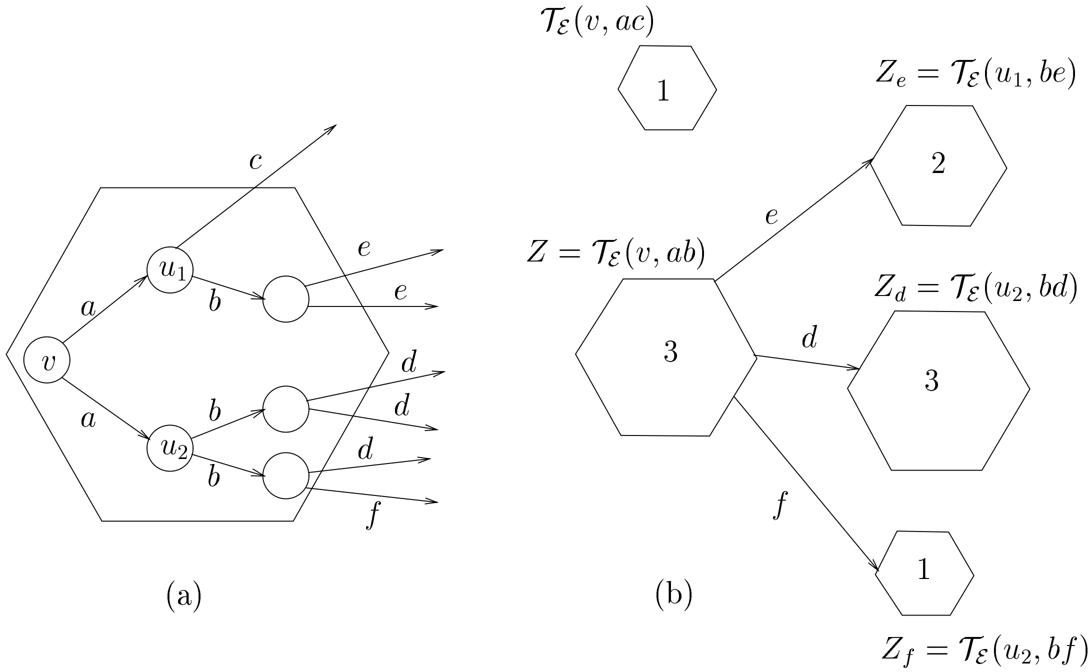

Figure 5: State Z in $G_{\mathcal{E}}$.

Lemma 4.3 *For $i = 0, 1, \ldots, \mathsf{a}$, the vector $\boldsymbol{c}^{(i)}$ is a strictly positive $(A_{G_{\mathcal{E}}^{(i)}}, n)$-integral eigenvector and $\|\boldsymbol{c}^{(i)}\| \le n^{\mathsf{a}-i}$.*

Proof. Let $Z = \mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ be a state in $G_{\mathcal{E}}^{(i)}$. All the paths of $\mathcal{E}$ within Z may differ only on their last $\mathsf{a}-i$ edges. Since the out-degree of each state in $\mathcal{E}$ is n, there can be at most $n^{\mathsf{a}-i}$ distinct paths in Z. Hence, $\|\boldsymbol{c}^{(i)}\| \le n^{\mathsf{a}-i}$. Also, from $(\boldsymbol{c}^{(i)})_Z = |Z|$, where Z represents a nonempty set of paths, it follows that $\boldsymbol{c}^{(i)}$ is strictly positive.

The rest of the proof is similar to an analogous result in [14]. By (1), the terminal states of the outgoing edges of Z in $G_{\mathcal{E}}^{(i)}$ are $Z_{b,e} = \mathcal{T}_{\mathcal{E}}((\pi e)^*, (\boldsymbol{w}b)^*)$, where b ranges over the symbols for which $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w}b) \ne \emptyset$ and e ranges over the edges in $\mathcal{E}$ labeled by the first symbol of $\boldsymbol{w}$ such that πe is a path in $\mathcal{E}$. Therefore,

$$(A_{G_{\mathcal{E}}^{(i)}} \boldsymbol{c}^{(i)})_Z = \sum_{b,e} |Z_{b,e}| = \sum_{b,e} |\mathcal{T}_{\mathcal{E}}((\pi e)^*, (\boldsymbol{w}b)^*)| = \sum_{b,e} |\mathcal{T}_{\mathcal{E}}(\pi e, (\boldsymbol{w}b)^*)| = \sum_{b} |\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w}b)| \,,$$

where the third equality holds since the paths in $\mathcal{T}_{\mathcal{E}}(\pi e, (\boldsymbol{w}b)^*)$ are of length $\mathsf{a}+1$ and thus share their first edge, and the last equality follows from taking the summation on e over all the edges that generate the first symbol in $\boldsymbol{w}b$.

On the other hand, since the out-degree of each state in $\mathcal{E}$ is n, we have

$$\sum_{b} |\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w}b)| = \Big|\bigcup_{b} \mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w}b)\Big| = n \cdot |\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})| = n \cdot |Z| = n \cdot (\boldsymbol{c}^{(i)})_Z \,.$$

Hence, $A_{G_{\mathcal{E}}^{(i)}}\boldsymbol{c}^{(i)} = n\boldsymbol{c}^{(i)}$. ■

Lemma 4.4 *For $i = 1, 2, \ldots, \mathsf{a}$, the graph $G_{\mathcal{E}}^{(i)}$ can be obtained from $G_{\mathcal{E}}^{(i-1)}$ by one round of $\boldsymbol{c}^{(i-1)}$-consistent splitting in which each state of $G_{\mathcal{E}}^{(i-1)}$ is split into no more than n states. The induced approximate eigenvector is $\boldsymbol{c}^{(i)}$.*

Proof. The basic idea is that in the transition from $G_{\mathcal{E}}^{(i-1)}$ to $G_{\mathcal{E}}^{(i)}$, we trade words with paths: the word portion $\boldsymbol{w}$ in a state $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ becomes shorter, whereas π becomes longer.

Next we specify the round of splitting that produces $G_{\mathcal{E}}^{(i)}$ out of $G_{\mathcal{E}}^{(i-1)}$. For a state $Z = \mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ in $G_{\mathcal{E}}^{(i-1)}$, define $E_{\mathcal{E}}(\pi, \boldsymbol{w})$ to be the set of all edges of $\mathcal{E}$ that appear as an ith edge in at least one path in Z; note that the elements of $E_{\mathcal{E}}(\pi, \boldsymbol{w})$ are outgoing edges of the terminal state of π in $\mathcal{E}$ that are all labeled by the first symbol of $\boldsymbol{w}$, and, so, $|E_{\mathcal{E}}(\pi, \boldsymbol{w})| \le n$.

A state $Z = \mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w})$ in $G_{\mathcal{E}}^{(i-1)}$ splits into the following $|E_{\mathcal{E}}(\pi, \boldsymbol{w})| \le n$ descendant states

$$Y_e = \mathcal{T}_{\mathcal{E}}(\pi e, \boldsymbol{w}^*) \ , \quad e \in E_{\mathcal{E}}(\pi, \boldsymbol{w})$$

in $G_{\mathcal{E}}^{(i)}$. The outgoing edges of Z in $G_{\mathcal{E}}^{(i-1)}$ that Y_e 'inherits' are those of the form

$$\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w}) \xrightarrow{b} \mathcal{T}_{\mathcal{E}}((\pi e)^*, \boldsymbol{w}^* b) \ .$$

This distribution of edges among the descendant states of Z forms a partition. The outgoing edges of Y_e in $G_{\mathcal{E}}^{(i)}$ thus have the form

$$\mathcal{T}_{\mathcal{E}}(\pi e, \boldsymbol{w}^*) \xrightarrow{b} \mathcal{T}_{\mathcal{E}}((\pi e)^* e', (\boldsymbol{w}^* b)^*) \ ,$$

where e' ranges over all edges in $E_{\mathcal{E}}((\pi e)^*, \boldsymbol{w}^* b)$ (see (1)).

The descendant states Y_e define a partition of the set of paths in Z; so,

$$\sum_{Y_e} (\boldsymbol{c}^{(i)})_{Y_e} = \sum_{Y_e} |Y_e| = |Z| = (\boldsymbol{c}^{(i-1)})_Z \ .$$

As $\boldsymbol{c}^{(i)}$ is an $(A_{G_{\mathcal{E}}^{(i)}}, n)$-integral eigenvector, we get that the round of splitting of $G_{\mathcal{E}}^{(i-1)}$ is $\boldsymbol{c}^{(i-1)}$-consistent and induces the integral eigenvector $\boldsymbol{c}^{(i)}$. ■

Figure 6 depicts the splitting of a state $Z \in V(G_{\mathcal{E}})$ in the first round, where $n = \mathsf{a} = 2$. The dashed edges indicate the partition of the outgoing edges of Z among its descendant states.

Lemmas 4.5 and 4.6 below deal with the two extreme cases $i = 0$ and $i = \mathsf{a}$.

Lemma 4.5 *The graph $G_{\mathcal{E}} = G_{\mathcal{E}}^{(0)}$ is an irreducible deterministic presentation of $S_{\mathcal{E}}$.*

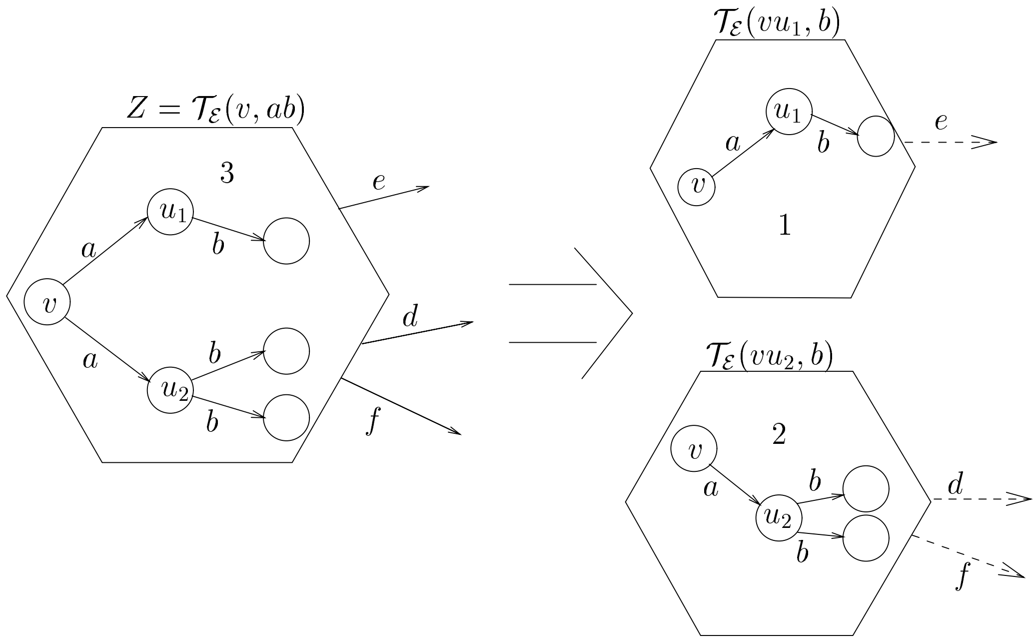

Figure 6: Splitting of state Z in $G_{\mathcal{E}}$.

Proof. Recall that all the paths in $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w}b)$ share the same first edge, e_1. Now, when $i = 0$, each path π in (1) is empty and stands for the initial state of e_1. Hence, for every symbol b such that $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{w}b) \neq \emptyset$, the edge e in (1) takes only the value e_1, implying that $G_{\mathcal{E}}$ is deterministic.

Let $\boldsymbol{w} = w_1 w_2 \ldots w_\ell$ be a word of length $\ell \geq \mathsf{a}$. There is a path

$$u_0 \xrightarrow{w_1} u_1 \xrightarrow{w_2} \ldots \xrightarrow{w_\ell} u_\ell$$

for some states $u_0, u_1, \ldots, u_\ell$ in $\mathcal{E}$ if and only if there is a path

$$\mathcal{T}_{\mathcal{E}}(u_0, w_1 w_2 \ldots w_{\mathsf{a}}) \xrightarrow{w_{\mathsf{a}+1}} \mathcal{T}_{\mathcal{E}}(u_1, w_2 w_3 \ldots w_{\mathsf{a}+1}) \xrightarrow{w_{\mathsf{a}+2}} \ldots \xrightarrow{w_\ell} \mathcal{T}_{\mathcal{E}}(u_{\ell-\mathsf{a}}, w_{\ell-\mathsf{a}+1} w_{\ell-\mathsf{a}+2} \ldots w_\ell)$$

(for the same states $u_0, u_1, \ldots, u_{\ell-\mathsf{a}}$) in $G_{\mathcal{E}}$. This, in turn, implies that $S(\mathcal{E}) = S(G_{\mathcal{E}})$ and that $G_{\mathcal{E}}$ is irreducible. ∎

We remark that the graph $G_{\mathcal{E}}$ is similar to the determinizing graph construction used in [14] and [4], except that the states therein are the sets of *terminal* states of paths in $\mathcal{E}$ that start in a given state and generate a given word $\boldsymbol{w}$ of any length. In our construction, the states in $G_{\mathcal{E}}$ are defined as the sets of the paths themselves, and the length of $\boldsymbol{w}$ equals the anticipation a.

Lemma 4.6 *The graph $G_{\mathcal{E}}^{(\mathsf{a})}$ is an $(S_{\mathcal{E}}, n)$-encoder with $\mathcal{A}(G_{\mathcal{E}}^{(\mathsf{a})}) = \mathcal{A}(\mathcal{E}) = \mathsf{a}$. Furthermore, $G_{\mathcal{E}}^{(\mathsf{a})}$ can be turned into the original encoder $\mathcal{E}$ by a reduction of strongly equivalent states.*

Proof. There is a one-to-one correspondence between the paths of length a in $\mathcal{E}$ and the states of $G_{\mathcal{E}}^{(\mathsf{a})}$: each path π of length a in $\mathcal{E}$ corresponds to a state $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{\epsilon})$, the latter set consisting of only one element, namely π. Furthermore, by (1), for every edge e labeled b that extends π to a path πe in $\mathcal{E}$ there is an edge

$$\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{\epsilon}) \xrightarrow{b} \mathcal{T}_{\mathcal{E}}((\pi e)^*, \boldsymbol{\epsilon})$$

in $G_{\mathcal{E}}^{(\mathsf{a})}$. The out-degree of a state $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{\epsilon})$ in $G_{\mathcal{E}}^{(\mathsf{a})}$ equals the out-degree, n, of the terminal state of π in $\mathcal{E}$.

It is easy to verify that states $\mathcal{T}_{\mathcal{E}}(\pi, \boldsymbol{\epsilon})$ that correspond to paths π with the same terminal state u of $\mathcal{E}$ are all strongly equivalent. By reducing those states, we readily obtain the graph $\mathcal{E}$.

By Lemma 4.4, the graph $G_{\mathcal{E}}^{(\mathsf{a})}$ can be obtained from the deterministic graph $G_{\mathcal{E}}$ by a rounds of splitting. Hence, $G_{\mathcal{E}}^{(\mathsf{a})}$ is an $(S_{\mathcal{E}}, n)$-encoder with $\mathcal{A}(G_{\mathcal{E}}^{(\mathsf{a})}) \le \mathsf{a}$. And by Proposition 2.6, this inequality is tight. ■

Proof of Proposition 4.1. Combine Lemmas 4.3–4.6. ■

Proof of Proposition 4.2. Follows from Proposition 4.1 and Lemma 2.3. ■

5 Proof of main results

We provide in this section proofs of Theorems 3.1 and 3.2 based on Propositions 4.1 and 4.2. Most of the section will include definitions and lemmas that will lead to the proof of Theorem 3.1. The proof of Theorem 3.2 will be given at the end of the section.

Let $\mathcal{E}$ be an irreducible (S, n)-encoder with anticipation $\mathsf{a} < \infty$ and let $S_{\mathcal{E}} = S(\mathcal{E})$. Also, let $G_{\mathcal{E}}$ be obtained from $\mathcal{E}$ as described in Section 4. We construct next a graph $\mathcal{G}$ from $G_{\mathcal{E}}$ and the Shannon cover G_S through intermediate graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, as described in Figure 7 ($\mathcal{G}$ will be the graph claimed in Theorem 3.1).

The idea behind the construction of $\mathcal{G}$ is as follows: On the one hand, $\mathcal{G}$ should be an irreducible deterministic presentation of S that satisfies $\{\mathcal{F}_{\mathcal{G}}(u) : u \in V(\mathcal{G})\} = \{\mathcal{F}_{G_S}(v) : v \in V(G_S)\}$. On the other hand, we want the states of $\mathcal{G}$ to mimic, in some sense, the splitting process of the states in $G_{\mathcal{E}}$, as indicated in Proposition 4.1.

The graph $\mathcal{G}$ is thus constructed so that every state (v', v) is equivalent to the state $v \in G_S$ and, at the same time, (v', v) can mimic the splitting of $v' \in G_{\mathcal{E}}$. Since there may be some states $v \in G_S$ for which there is no $v' \in V(G_{\mathcal{E}})$ such that $v' \preceq v$, we add in Step 3 the states $\{(\phi, v) : v \in G_S\}$. Steps 2 and 4 guarantee the irreducibility of $\mathcal{G}$. Following is a formal analysis of the construction.

We first verify that $\mathcal{G}_1$ is well-defined, namely, that $V(\mathcal{G}_1) \ne \emptyset$. Recall from Lemma 4.5 that $G_{\mathcal{E}}$ is an irreducible deterministic presentation of $S_{\mathcal{E}}$. Hence, by Proposition 2.1(iii),

1. Let $\mathcal{G}_1$ be the graph defined as follows. The set of states of $\mathcal{G}_1$ is given by

$$V(\mathcal{G}_1) = \{(v', v) \, : \, v' \in V(G_\mathcal{E}), v \in V(G_S), v' \preceq v\} \, ,$$

and $\mathcal{G}_1$ is endowed with an edge $(u', u) \stackrel{b}{\to} (v', v)$ if and only if there are edges $u' \stackrel{b}{\to} v'$ in $G_\mathcal{E}$ and $u \stackrel{b}{\to} v$ in G_S.

2. Let $\mathcal{G}_2$ be an irreducible sink of $\mathcal{G}_1$.

3. Let $\mathcal{G}_3$ be the graph obtained from $\mathcal{G}_2$ by adding the following set of states

$$\{(\phi, v) \, : \, v \in V(G_S)\}$$

and the following edges:

   (a) For $(u', u) \in V(\mathcal{G}_2)$, if $u \stackrel{b}{\to} v$ is an edge in G_S for some state v but there is no outgoing edge of u' labeled b in $G_\mathcal{E}$, then endow $\mathcal{G}_3$ with an edge $(u', u) \stackrel{b}{\to} (\phi, v)$.

   (b) Endow (ϕ, u) with an outgoing edge labeled b in $\mathcal{G}_3$ per every symbol b for which there is an edge $u \stackrel{b}{\to} v$ in G_S. The terminal state of each such edge in $\mathcal{G}_3$ is (v', v), where v' is the smallest (according to some lexicographic ordering) state in $\mathcal{G}_1$ for which (v', v) is a state in $\mathcal{G}_3$; if no such state v' exists, set $v' = \phi$.

4. To obtain $\mathcal{G}$ from $\mathcal{G}_3$ use the following procedure:
If there is a state (ϕ, v) that has no incoming edges, delete (ϕ, v) and its outgoing edges.
Continue deleting until no such states exist.

Figure 7: Construction of $\mathcal{G}$.

for every state u' in $G_\mathcal{E}$ there is at least one state u in G_S such that $u' \preceq u$ (that is, $\mathcal{F}_{G_\mathcal{E}}(u') \subseteq \mathcal{F}_{G_S}(u)$). Note that if there exist edges $u' \stackrel{b}{\to} v'$ and $u \stackrel{b}{\to} v$ in $G_\mathcal{E}$ and G_S, respectively, then $v' \preceq v$ and, so, (v', v) is a state in $\mathcal{G}_1$. From this we obtain by induction on ℓ that $\mathcal{F}_{\mathcal{G}_1}(u', u)$ and $\mathcal{F}_{G_\mathcal{E}}(u')$ contain the same set of words of length ℓ. Therefore,

$$\mathcal{F}_{\mathcal{G}_1}(u', u) = \mathcal{F}_{G_\mathcal{E}}(u') \tag{2}$$

for every $(u', u) \in V(\mathcal{G}_1)$. (One can also show that $S(\mathcal{G}_1) = S_\mathcal{E}$. We skip the proof, as this property is not required in the sequel.)

Lemma 5.1 *The graph $\mathcal{G}_2$ is an irreducible deterministic presentation of $S_\mathcal{E}$ and is an expansion of $G_\mathcal{E}$.*

Proof. Since $\mathcal{G}_2$ is an irreducible sink of $\mathcal{G}_1$, it follows from (2) that $\mathcal{F}_{\mathcal{G}_2}(u', u) = \mathcal{F}_{G_\mathcal{E}}(u')$ for every $(u', u) \in V(\mathcal{G}_2)$. From the irreducibility of $G_\mathcal{E}$ we thus obtain $S(\mathcal{G}_2) = S(G_\mathcal{E}) = S_\mathcal{E}$.

To show that $\mathcal{G}_2$ is an expansion of $G_\mathcal{E}$ it suffices to prove that for every $u' \in V(G_\mathcal{E})$ there is a state $u \in V(G_S)$ such that $(u', u) \in V(\mathcal{G}_2)$. Indeed, let (v', v) be a state in $V(\mathcal{G}_2)$. As $G_\mathcal{E}$ is irreducible, there is a path from v' to u' in $G_\mathcal{E}$. A respective path must exist in

$\mathcal{G}_1$ from (v', v) to (u', u) for some $u \in V(G_S)$, and this path is also a path of $\mathcal{G}_2$, as $\mathcal{G}_2$ is an irreducible sink of $\mathcal{G}_1$. ■

Let $\boldsymbol{c} = \boldsymbol{c}_{\mathcal{E}}$ be an $(A_{G_{\mathcal{E}}}, n)$-integral eigenvector of $G_{\mathcal{E}}$ as in Proposition 4.1. We define $\boldsymbol{c}_2 = [(\boldsymbol{c}_2)_{(v',v)}]_{(v',v)\in V(\mathcal{G}_2)}$ to be the respective expansion $(A_{\mathcal{G}_2}, n)$-integral eigenvector (see Section 2.4), i.e.,

$$(\boldsymbol{c}_2)_{(v',v)} = (\boldsymbol{c})_{v'} \ . \qquad (3)$$

Note that by Proposition 4.1(i) it follows that $\boldsymbol{c}_2$ is strictly positive.

Lemma 5.2 *The graph $\mathcal{G}_2$ and the $(A_{\mathcal{G}_2}, n)$-integral eigenvector $\boldsymbol{c}_2$ satisfy Theorem 3.1 with respect to the constraint $S_{\mathcal{E}}$.*

Proof. Lemma 5.1 implies that $\mathcal{G}_2$ is irreducible. Conditions (i), (iii), and (iv) of Theorem 3.1 follow directly from Proposition 2.8 (and its proof) and Proposition 4.1. Now, Propositions 2.8 and 4.1 imply that $\mathcal{G}_2$ can be n-fully split in at most a rounds consistently with $\boldsymbol{c}_2$ to obtain an $(S_{\mathcal{E}}, n)$-encoder $\mathcal{E}_2$. The number of rounds bounds $\mathcal{A}(\mathcal{E}_2)$ from above; on the other hand, from Propositions 2.6 and 4.1 we obtain the lower bound $\mathcal{A}(\mathcal{E}_2) \geq \mathsf{a}$. Thus, $\mathcal{A}(\mathcal{E}_2) = \mathsf{a}$, thereby yielding condition (ii) of Theorem 3.1.

As for condition (v), it follows from Proposition 2.8 and Lemma 5.1 that $\mathcal{E}_2$ is an expansion of the encoder $\mathcal{E}'$ obtained from $G_{\mathcal{E}}$ in Proposition 4.1(ii). On the other hand, by Proposition 4.1(v), the encoder $\mathcal{E}'$ is an expansion of the original encoder $\mathcal{E}$; therefore, $\mathcal{E}_2$ is an expansion of $\mathcal{E}$. Equivalently, $\mathcal{E}_2$ can turn into $\mathcal{E}$ by a reduction of strongly equivalent states. ■

In analogy with (2), it is easy to verify the equality

$$\mathcal{F}_{\mathcal{G}_3}(u', u) = \mathcal{F}_{G_S}(u)$$

for every $(u', u) \in V(\mathcal{G}_3)$. This, in turn, implies the next lemma.

Lemma 5.3 $S(\mathcal{G}_3) = S$.

Lemma 5.4 *The graph $\mathcal{G}$ is an irreducible deterministic presentation of S.*

Proof. We first show that $\mathcal{G}$ is irreducible. Since $\mathcal{G}_2$ is an irreducible subgraph of $\mathcal{G}$, it suffices to show that for each state $(\phi, u) \in V(\mathcal{G}) \setminus V(\mathcal{G}_2)$ there is a path in $\mathcal{G}$ from (ϕ, u) to some state in $V(\mathcal{G}_2)$ and a path from $V(\mathcal{G}_2)$ back to (ϕ, u).

Fix $(\phi, u) \in V(\mathcal{G}) \setminus V(\mathcal{G}_2)$ and let (v', v) be a state in $V(\mathcal{G}_2)$. There is a path from u to v in G_S and a respective path in $\mathcal{G}$ that starts in (ϕ, u) and has the form

$$(\phi, u) \to (u'_1, u_1) \to (u'_2, u_2) \to \ldots \to (u'_\ell, u_\ell) = (u'_\ell, v) \ .$$

If $u'_i \neq \phi$ for some $i < \ell$, we are done. Otherwise, if $u_i = \phi$ for every $i < \ell$, then u'_ℓ must be v' or some other state $v'' \in V(G_\mathcal{E})$ that is dominated by v. We thus conclude that there is a path in $\mathcal{G}$ from (ϕ, u) to some state in $V(\mathcal{G}_2)$.

Recall that by construction, the state (ϕ, u) must have an incoming edge in $\mathcal{G}$. If the initial state of this edge belongs to $V(\mathcal{G}_2)$, then we are done. Otherwise, that initial state is in $V(\mathcal{G}) \setminus V(\mathcal{G}_2)$, which means that $(u', u) \in V(\mathcal{G})$ implies $u' = \phi$. Let (v', v) be a state in $V(\mathcal{G}_2)$. There is a path in G_S from v to u, and the respective path in $\mathcal{G}$ terminates in $(u', u) = (\phi, u)$. Hence, there is a path in $\mathcal{G}$ from every state in $V(\mathcal{G}_2)$ to (ϕ, u), thereby establishing the irreducibility of $\mathcal{G}$.

It follows by construction that the irreducible components of $\mathcal{G}_3$ are given by the irreducible sink $\mathcal{G}$ and (possibly) isolated states $(\phi, u) \in V(\mathcal{G}_3) \setminus V(\mathcal{G})$. Hence,

$$\mathcal{F}_\mathcal{G}(u', u) = \mathcal{F}_{\mathcal{G}_3}(u', u) = \mathcal{F}_{G_S}(u)$$

for every $(u', u) \in V(\mathcal{G})$. This implies that $S(\mathcal{G}) = S$. ∎

By Proposition 2.7 it thus follows that $\mathcal{G}$ is an expansion of G_S.

Proof of Theorem 3.1. Let $\mathcal{G}_2$ and $\mathcal{G}$ be defined by Figure 7 and let $\boldsymbol{c}_2$ be the $(A_{\mathcal{G}_2}, n)$-integral eigenvector defined by (3). Define the vector $\boldsymbol{x}$ by

$$(\boldsymbol{x})_{(v',v)} = \begin{cases} (\boldsymbol{c}_2)_{(v',v)} & \text{if } (v', v) \in V(\mathcal{G}_2) \\ 0 & \text{otherwise} \end{cases} , \quad (v', v) \in V(\mathcal{G}) .$$

By Lemma 5.4 the graph $\mathcal{G}$ is irreducible. Next we show that $\boldsymbol{x}$ is an $(A_\mathcal{G}, n)$-approximate eigenvector. First, $\boldsymbol{x}$ is nonzero since $\boldsymbol{c}_2$ is strictly positive. Now, for every $(u', u) \in V(\mathcal{G}_2)$ we have

$$(A_\mathcal{G}\boldsymbol{x})_{(u',u)} = (A_{\mathcal{G}_2}\boldsymbol{c}_2)_{(u',u)} + 0 = n \cdot (\boldsymbol{c}_2)_{(u',u)} = n \cdot (\boldsymbol{x})_{(u',u)} ,$$

and for $(\phi, u) \in V(\mathcal{G}) \setminus V(\mathcal{G}_2)$ we get

$$(A_\mathcal{G}\boldsymbol{x})_{(\phi,u)} \geq 0 = n \cdot (\boldsymbol{x})_{(\phi,u)} .$$

Hence, $\boldsymbol{x}$ is indeed an $(A_\mathcal{G}, n)$-approximate eigenvector.

Finally, we verify that $\mathcal{G}$ and $\boldsymbol{x}$ satisfy conditions (i)–(v) of Theorem 3.1. Indeed, after removing the zero weight states from $\mathcal{G}$, we are left with the graph $\mathcal{G}_2$ and the respective (integral) eigenvector $\boldsymbol{c}_2$. The result now follows from Lemma 5.2. ∎

Proof of Theorem 3.2. Let $G_\mathcal{E}$ and $\boldsymbol{c} = \boldsymbol{c}_\mathcal{E}$ be as in Proposition 4.1, let G be any irreducible deterministic presentation of S, and let k be any positive integer in the range $1 \leq k \leq \mathsf{a}$. We define an (A_G, n)-approximate eigenvector $\boldsymbol{x}$ from $\boldsymbol{c}$ and show how a $\boldsymbol{c}$-consistent round of splitting of $G_\mathcal{E}^k$ that satisfies the conditions of Proposition 4.2 can be transformed into an $\boldsymbol{x}$-consistent round of splitting of G^k that satisfies the conditions of Theorem 3.2.

For a state $u \in V(G)$ define the set $W(u) = \{Y \in V(G_{\mathcal{E}}) : Y \preceq u\}$ and let $Z(u)$ be an element in $V(G_{\mathcal{E}}) \cup \{\phi\}$ defined as follows: if $|W(u)| > 0$ then $Z(u)$ is a particular state in $W(u)$ such that $(\boldsymbol{c})_{Z(u)} \geq (\boldsymbol{c})_Y$ for every $Y \in W(u)$; otherwise (if $|W(u)| = 0$) set $Z(u) = \phi$. By Proposition 2.1, there is at least one state $u \in V(G_{\mathcal{E}})$ for which $|W(u)| > 0$, in which case $Z(u) \neq \phi$.

Define the vector $\boldsymbol{x}$ by

$$(\boldsymbol{x})_u = (\boldsymbol{c})_{Z(u)} , \quad u \in V(G) ,$$

where $(\boldsymbol{c})_\phi \equiv 0$. By part (c) of the proof of Theorem 3 in [14] it follows that $\boldsymbol{x}$ is an (A_G, n)-approximate eigenvector (note that Proposition 4.1(i) implies that $\boldsymbol{x}$ is nonzero).

Let $\boldsymbol{c}'$ be the induced integral eigenvector obtained from the $\boldsymbol{c}$-consistent round of splitting of $G_{\mathcal{E}}^k$ as in Proposition 4.2. We now show a respective $\boldsymbol{x}$-consistent round of splitting of G^k that produces a graph H and an induced (A_H^k, n^k)-approximate eigenvector $\boldsymbol{x}'$. States u in G^k for which $Z(u) = \phi$ have zero weight and are therefore deleted. An irreducible sink $\tilde{G}$ of the resultant graph is then selected. Suppose now that u is a state of $\tilde{G}$ ($Z = Z(u) \neq \phi$), and let $\{E_Z^{(r)}\}_{r=1}^{N(Z)}$ be the $\boldsymbol{c}$-consistent partition of the outgoing edges of Z in $G_{\mathcal{E}}^k$ that defines the splitting of Z to produce the descendant states $\{(Z(u))^{(r)}\}_{r=1}^{N(Z)} = \{Z^{(r)}\}_{r=1}^{N(Z)}$ with respective weights as defined by $\boldsymbol{c}'$; recall that by Proposition 4.2 we have $N(Z) \leq n^k$ and $\|\boldsymbol{c}'\| \leq n^{\mathsf{a}-k}$.

Since both $\tilde{G}$ and $G_{\mathcal{E}}^k$ are deterministic and $Z \preceq u$, we can partition the set of outgoing edges of u in $\tilde{G}$ into $\{E_u^{(r)}\}_{r=1}^{N(u)}$, where $N(u) = N(Z)$ and the set of labels of $E_Z^{(r)}$ is a subset of the set of labels of $E_u^{(r)}$ for $r = 1, 2, \ldots, N(u)$. The partition $\{E_u^{(r)}\}_{i=1}^{N(u)}$, in turn, defines a splitting of u in $\tilde{G}$ into descendant states $\{u^{(r)}\}_{r=1}^{N(u)}$ in H where $N(u) \leq n^k$.

Let $\boldsymbol{x}'$ be defined by

$$(\boldsymbol{x}')_{u^{(r)}} = (\boldsymbol{c}')_{(Z(u))^{(r)}} , \quad u^{(r)} \in V(H) .$$

Clearly, we have

$$(\boldsymbol{x})_u = (\boldsymbol{c})_{Z(u)} = \sum_{r=1}^{N(Z(u))} (\boldsymbol{c}')_{(Z(u))^{(r)}} = \sum_{r=1}^{N(u)} (\boldsymbol{x}')_{u^{(r)}}$$

for every $u \in V(G)$. Furthermore, $\|\boldsymbol{x}'\| = \|\boldsymbol{c}'\| \leq n^{\mathsf{a}-k}$.

Next we verify that $\boldsymbol{x}'$ is an (A_H, n^k)-approximate eigenvector; this, in turn, will imply that the round of splitting of G^k is indeed $\boldsymbol{x}$-consistent and $\boldsymbol{x}'$ is the induced approximate eigenvector. Indeed,

$$\begin{aligned}(A_H \boldsymbol{x}')_{u^{(r)}} = \sum_{e \in E_u^{(r)}} (\boldsymbol{x})_{\tau_{G^k}(e)} &= \sum_{e \in E_u^{(r)}} (\boldsymbol{c})_{Z(\tau_{G^k}(e))} \\ &\geq \sum_{e' \in E_{Z(u)}^{(r)}} (\boldsymbol{c})_{\tau_{G_{\mathcal{E}}^k}(e')} = n^k \cdot (\boldsymbol{c}')_{(Z(u))^{(r)}} = n^k \cdot (\boldsymbol{x}')_{u^{(r)}} ,\end{aligned}$$

where the inequality follows from observing that if e and e' are edges with the same label in $E_u^{(r)}$ and $E_{Z(u)}^{(r)}$, respectively, then $\tau_{G_{\mathcal{E}}^k}(e') \preceq \tau_{G^k}(e)$ and, so, $(\boldsymbol{c})_{\tau_{G_{\mathcal{E}}^k}(e')} \leq (\boldsymbol{c})_{Z(\tau_{G^k}(e))}$. ∎

6 Computing lower bounds

6.1 The general case

Theorem 3.1 provides a *necessary and sufficient* condition on the existence of (S, n)-encoders with anticipation a. Next, we would like to claim that the following problem is decidable: Given a constraint S and integers n and a, does there exist an (S, n)-encoder with anticipation a? To this end, it suffices to show an upper bound on the size of the deterministic presentation $\mathcal{G}$ in Theorem 3.1. Such an upper bound can be stated if we slightly weaken the theorem, as follows.

Theorem 6.1 *Let S be an irreducible constraint and let n be a positive integer where $\mathsf{cap}(S) \geq \log_2 n$. Denote by Δ the largest out-degree of any state in the Shannon cover G_S of S. Suppose there exists some irreducible (S, n)-encoder $\mathcal{E}$ with $\mathcal{A}(\mathcal{E}) = \mathsf{a} < \infty$. Then there exists an irreducible deterministic (not necessarily reduced) presentation $\mathcal{G}$ of S with at most*

$$|V(G_S)| \cdot (n^{\mathsf{a}} + 1)^{(\Delta+1)^{\mathsf{a}}} \cdot n^{((\Delta+1)^{\mathsf{a}}-1)\cdot(n^{\mathsf{a}+1}/(\Delta-1))} \qquad (4)$$

states and an $(A_{\mathcal{G}}, n)$-approximate eigenvector $\boldsymbol{x}$ that satisfy the following:

(i) $\|\boldsymbol{x}\| \leq n^{\mathsf{a}}$.

(ii) The graph $\mathcal{G}$ can be n-fully split consistently with $\boldsymbol{x}$ in a rounds of splitting. After deleting excess edges, an (S, n)-encoder $\mathcal{E}_{\mathcal{G}}$ is obtained with $\mathcal{A}(\mathcal{E}_{\mathcal{G}}) \leq \mathsf{a}$.

(iii) In each of the splitting rounds, every state is split into at most n states.

(iv) In the ith round, the induced approximate eigenvector $\boldsymbol{x}^{(i)}$ satisfies $\|\boldsymbol{x}^{(i)}\| \leq n^{\mathsf{a}-i}$.

(v) Every state in $\mathcal{E}_{\mathcal{G}}$ is equivalent to a state in $\mathcal{E}$.

The proof of Theorem 6.1 can be found in [16]. Note that Theorem 6.1 differs from Theorem 3.1 in that the states of $\mathcal{E}_{\mathcal{G}}$ are equivalent—but not necessarily strongly equivalent—to the states of $\mathcal{E}$, and that $\mathcal{A}(\mathcal{E}_{\mathcal{G}})$ might be smaller than a. It is also shown in [16] that if condition (v) is removed from Theorem 6.1, then the upper bound (4) can be improved to

$$|V(G_S)| \cdot \left(1 + n^{\mathsf{a}+(\mathsf{a}-1)n^{\mathsf{a}}}\right) .$$

A similar decidability result was obtained in [4] by computing an upper bound on the smallest number of states in any (S, n)-encoder with anticipation a. Yet, the upper bounds in [4] and herein are too big to imply any *efficient* algorithm for deciding whether an (S, n)-encoder with anticipation a exists. In particular, they do not imply an efficient algorithm for computing tight bounds on the anticipation of any (S, n)-encoder for *every* given S and n. However, in what follows we demonstrate how Theorems 3.1 and 3.2 can still be applied to obtain lower bounds on the anticipation of encoders at given rates for specific constraints.

6.2 Specific examples

In the examples below, we study three constraints, for which we obtain tight lower bounds. Our results are summarized in Table 1. The table also lists references to encoders that attain our bounds. For comparison, in all three examples, the lower bounds of [14] are smaller by 1 than the figures in the table.

Example	Constraint	Capacity	Rate	Lower bound	Attaining encoders
6.1	$(1,7)$-RLL	0.6793	$2:3$	2	[1], [20]
6.2	$(2,7)$-RLL	0.5174	$1:2$	3	[8] (Figure 2)
6.3, B.1	$(2,18,2)$-RLL	0.4040	$2:5$	3	[19], [9]

Table 1: Tight bounds on the anticipation of encoders for several (d,k)-RLL constraints.

Both Theorem 3.1 and Theorem 3.2 can be used to obtain lower bounds. The general technique is a proof by contradiction: to show that the anticipation of any (S,n)-encoder must be greater than a, we assume that there is such an encoder with anticipation at most a and show that for *every* deterministic presentation $\mathcal{G}$ of S there is no $(A_{\mathcal{G}},n)$-approximate eigenvector $\boldsymbol{x}$ that satisfies the conditions of Theorem 3.1. Alternatively, we can show that for *a particular* deterministic presentation G of S (e.g., the Shannon cover G_S of S) and *a particular* value of k there is no (A_G,n)-approximate eigenvector $\boldsymbol{x}$ that satisfies the conditions of Theorem 3.2. Since we can fix G (and k) when applying Theorem 3.2, it will typically be simpler to obtain bounds from Theorem 3.2 than by using Theorem 3.1 (see Example 6.3 versus Example B.1).

As pointed out in Section 3, the previously known lower bounds are implied by our bounding technique: Theorem 2.4 is equivalent to Theorem 3.2(i), while Theorem 2.5 is equivalent to Theorem 3.2(ii) for the special case $k=\mathsf{a}$. Examples 6.1–6.3 demonstrate that our bounding technique is strictly stronger than Theorem 2.4, and Example 6.4 shows a strict improvement over Theorem 2.5.

The next lemma will allow us to overcome the difficulty of examining *all* the irreducible deterministic presentations $\mathcal{G}$ of S when using Theorem 3.1. Recall that by Proposition 2.1, for every state $u \in V(\mathcal{G})$ there exists a unique state $u' \in V(G_S)$ so that $\mathcal{F}_{\mathcal{G}}(u) = \mathcal{F}_{G_S}(u')$.

Lemma 6.2 *Let G_S be the Shannon cover of an irreducible constraint S and let $\mathcal{G}$ be an irreducible deterministic presentation of S. For every $(A_{\mathcal{G}},n)$-approximate eigenvector $\boldsymbol{x}$ there is an (A_{G_S},n)-approximate eigenvector $\boldsymbol{x}'$ such that $\|\boldsymbol{x}\| = \|\boldsymbol{x}'\|$ and*

$$(\boldsymbol{x})_u \le (\boldsymbol{x}')_{u'} \quad \textit{whenever} \quad \mathcal{F}_{\mathcal{G}}(u) = \mathcal{F}_{G_S}(u') \ .$$

Proof. Given an $(A_{\mathcal{G}},n)$-approximate eigenvector $\boldsymbol{x}$, define the vector $\boldsymbol{x}'$ by

$$(\boldsymbol{x}')_{u'} = \max_{u \in V(\mathcal{G})\,:\,\mathcal{F}_{\mathcal{G}}(u) = \mathcal{F}_{G_S}(u')} (\boldsymbol{x})_u \ , \quad u' \in V(G_S) \ . \tag{5}$$

By construction, $\|\boldsymbol{x}\| = \|\boldsymbol{x}'\|$ and $(\boldsymbol{x})_u \le (\boldsymbol{x}')_{u'}$ whenever $\mathcal{F}_{\mathcal{G}}(u) = \mathcal{F}_{G_S}(u')$. The proof that $\boldsymbol{x}'$ is an (A_{G_S}, n)-approximate eigenvector is contained in [14, p. 747] and is repeated here for the sake of completeness. For $u' \in V(G_S)$ let u be a state in $V(\mathcal{G})$ that attains the maximum in (5). We note that if $u \stackrel{a}{\to} v$ and $u' \stackrel{a}{\to} v'$ are edges in $\mathcal{G}$ and G_S, respectively, with the same label a, then $\mathcal{F}_{\mathcal{G}}(v) = \mathcal{F}_{G_S}(v')$ and, so, $(\boldsymbol{x}')_{v'} \ge (\boldsymbol{x})_v$. Hence,

$$(A_{G_S}\boldsymbol{x}')_{u'} \ge (A_{\mathcal{G}}\boldsymbol{x})_u \ge n \cdot (\boldsymbol{x})_u = n \cdot (\boldsymbol{x}')_{u'} \ ,$$

thereby implying that $\boldsymbol{x}'$ is an (A_{G_S}, n)-approximate eigenvector. ∎

The following example and Example B.1 are applications of Theorem 3.1 and Lemma 6.2 to obtain a lower bound on the anticipation.

Example 6.1 We show that every rate $2 : 3$ encoder for the $(1, 7)$-RLL constraint has anticipation at least 2. Let $S_{1,7}$ denote the $(1, 7)$-RLL constraint. From Figure 1 we see that the Shannon cover G_S of $S_{1,7}^3$ consists of eight states, $v_0, v_1, \ldots, v_7$. Figure 8 shows the set of outgoing edges and respective terminal states of each state in G_S. Using Franaszek's algorithm for computing approximate eigenvectors [15, p. 1671], we find that

$$[\,2\ 3\ 3\ 3\ 2\ 2\ 2\ 1\,]^\top \quad \text{and} \quad [\,2\ 3\ 3\ 3\ 2\ 2\ 2\ 0\,]^\top$$

are the only $(A_{G_S}, 4)$-approximate eigenvectors $\boldsymbol{x}'$ with $\|\boldsymbol{x}'\| \le 4$. For both vectors we have in fact $\|\boldsymbol{x}'\| = 3$, and the states u' with weight $(\boldsymbol{x}')_{u'} = 3$ are v_1, v_2, and v_3. Each of these three states has out-degree 5 in G_S.

Assume to the contrary that there exists an $(S_{1,7}^3, 2^2)$-encoder with anticipation $\mathsf{a} = 1$ and let $\mathcal{G}$ and $\boldsymbol{x}$ be as in Theorem 3.1 (here $n = 2^2 = 4$). In particular, from Theorem 3.1(i) we have $\|\boldsymbol{x}\| \le 4$. Since each of the two $(A_{G_S}, 4)$-approximate eigenvectors $\boldsymbol{x}'$ with $\|\boldsymbol{x}'\| \le 4$ satisfies $\|\boldsymbol{x}'\| = 3$, we obtain from Lemma 6.2 that $\|\boldsymbol{x}\| = 3$. Furthermore, it follows from the lemma that there is a state $u \in V(\mathcal{G})$ with weight $(\boldsymbol{x})_u = 3$ and u is equivalent to either v_1, v_2, or v_3 (i.e., $\mathcal{F}_{\mathcal{G}}(u) \in \{\mathcal{F}_{G_S}(v_1), \mathcal{F}_{G_S}(v_2), \mathcal{F}_{G_S}(v_3)\}$). In particular, the out-degree of u in $\mathcal{G}$ is 5.

By Theorem 3.1, the graph $\mathcal{G}$ is 4-fully split in one round consistently with $\boldsymbol{x}$. State u, having weight 3, is split in that round into three descendant states, each of weight 1, and the set of outgoing edges of u is partitioned among the descendant states. Since state u has five outgoing edges in $\mathcal{G}$, at least one of the descendant states of u inherits only *one* outgoing edge, say edge e. Now, in an $\boldsymbol{x}$-consistent splitting of state u we must have

$$(\boldsymbol{x})_{\tau_{\mathcal{G}}(e)} \ge n = 4 \ ,$$

but this contradicts the inequality $(\boldsymbol{x})_{\tau_{\mathcal{G}}(e)} \le \|\boldsymbol{x}\| = 3$. ∎

Our next three examples demonstrate how Theorem 3.2 can be applied to obtain lower bounds. Since that theorem holds for any irreducible deterministic presentation of the constraint S, we will typically apply the theorem to the Shannon cover G_S. As in the previous example, we show that every (S, n)-encoder must have anticipation greater than a by contradiction.

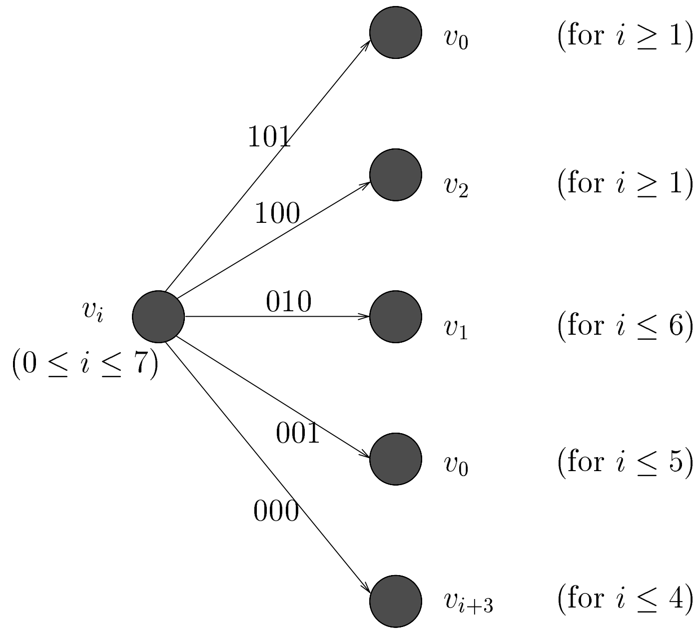

Figure 8: Outgoing edges of each state in the Shannon cover of $S_{1,7}^3$.

Example 6.2 We show that every rate $1:2$ encoder for the $(2,7)$-RLL constraint has anticipation at least 3. Let $S_{2,7}$ denote the $(2,7)$-RLL constraint. Figure 9 shows the set of outgoing edges and respective terminal states of each of the eight states, $v_0, v_1, \ldots, v_7$, in the Shannon cover G_S of $S_{2,7}^2$. By Franaszek's algorithm we find that every $(A_{G_S}, 2)$-approximate eigenvector $\boldsymbol{x}$ satisfies $\|\boldsymbol{x}\| \geq 4$, thereby implying that the anticipation of any $(S_{2,7}^2, 2)$-encoder is at least 2.

Assume to the contrary that there exists an $(S_{2,7}^2, 2)$-encoder with anticipation $\mathsf{a} = 2$. We apply Theorem 3.2 with $G = G_S$ and $k = 1$ and we let $\boldsymbol{x}$ be the $(A_{G_S}, 2)$-approximate eigenvector guaranteed by the theorem; in particular, by Theorem 3.2 we have $\|\boldsymbol{x}\| = 4$. Using Franaszek's algorithm we find that $\boldsymbol{x} \leq \boldsymbol{\xi} = [\,2\;3\;4\;4\;3\;3\;2\;1\,]^\top$. In particular, we must have $(\boldsymbol{x})_{v_2} = 4$ or $(\boldsymbol{x})_{v_3} = 4$. Figure 10 shows the outgoing edges of v_2 and v_3, where the weights of the states are determined by $\boldsymbol{\xi}$.

Suppose now that $(\boldsymbol{x}_{v_2}) = 4$; the case $(\boldsymbol{x})_{v_3} = 4$ is very similar. From Figure 10 and the inequality $\boldsymbol{x} \leq \boldsymbol{\xi}$ we obtain

$$3+2+3 = (\boldsymbol{\xi})_{v_4} + (\boldsymbol{\xi})_{v_0} + (\boldsymbol{\xi})_{v_1} \geq (\boldsymbol{x})_{v_4} + (\boldsymbol{x})_{v_0} + (\boldsymbol{x})_{v_1} \geq n \cdot (\boldsymbol{x})_{v_2} = 2 \cdot 4 = 8\;,$$

thereby forcing $(\boldsymbol{x})_v = (\boldsymbol{\xi})_v$ for $v \in \{v_4, v_0, v_1\}$. By Theorem 3.2(ii), state v_2 is split into two descendant states, each of weight 2. At least one of the descendant states of v_2 inherits a single edge e for which $\tau_{G_S}(e) < 4$. However, such a splitting is not $\boldsymbol{x}$-consistent. ■

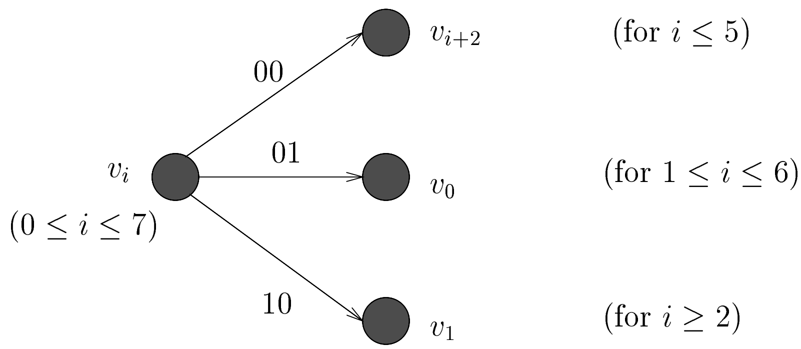

Figure 9: Outgoing edges of each state in the Shannon cover of $S^2_{2,7}$.

Figure 10: Outgoing edges of v_2 and v_3.

Example 6.3 We use Theorem 3.2 to prove the lower bound 3 on the anticipation of any rate $2:5$ encoder for the $(2,18,2)$-RLL constraint. Though this case is somewhat more complicated than previous examples, the lower bound can be obtained through Theorem 3.1 as well (see Appendix B).

Let $S_{2,18,2}$ denote the $(2,18,2)$-RLL constraint. The Shannon cover G_S of $S^5_{2,18,2}$ has nineteen states, $v_0, v_1, \ldots, v_{18}$, each having out-degree at most 5. Figure 11 shows the set of outgoing edges and respective terminal states of each state in G_S. Using Franaszek's algorithm we find that every $(A_{G_S}, 4)$-approximate eigenvector $\boldsymbol{x}$ satisfies $\|\boldsymbol{x}\| \ge 12$. This already rules out the possibility of having $(S^5_{2,18,2}, 2^2)$-encoders with anticipation 1.

Assume to the contrary that there exists an $(S^5_{2,18,2}, 4)$-encoder with anticipation $\mathsf{a} = 2$. We apply Theorem 3.2 with $G = G_S$ and $k = 2$ and we let $\boldsymbol{x}$ be the $(A_{G_S}, 4)$-approximate eigenvector guaranteed by the theorem. From Theorem 3.2(i) we have $\|\boldsymbol{x}\| \le 16$. For our choice of k, Theorem 3.2(ii) states that G^2_S can be 4^2-fully split in one round consistently with $\boldsymbol{x}$. We see from Figure 11 that each state in G^2_S has at most 21 outgoing edges.

We distinguish between two cases according to the value of $\|\boldsymbol{x}\|$. *Case 1:* $12 \le \|\boldsymbol{x}\| \le 15$.

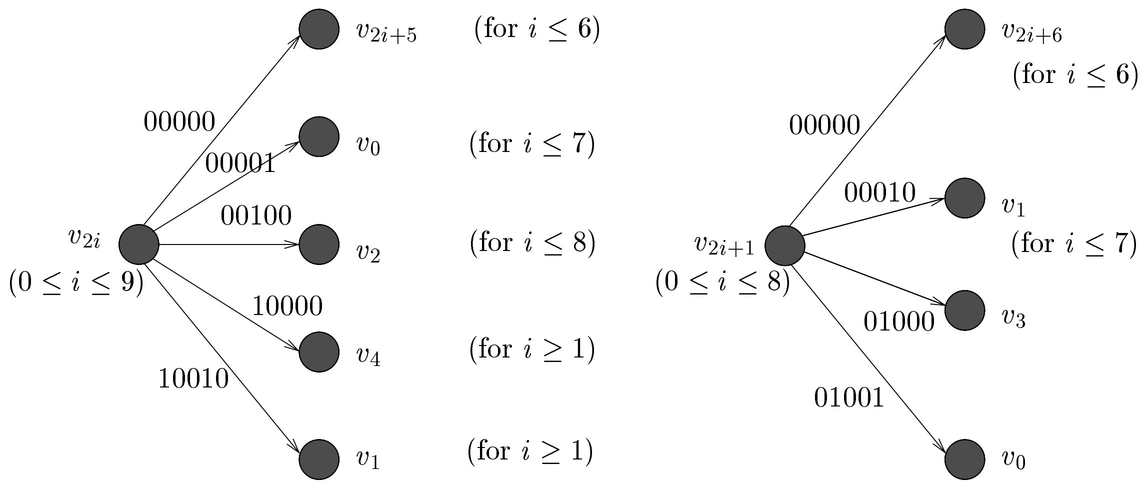

Figure 11: Outgoing edges of each state in the Shannon cover of $S^5_{2,18,2}$.

Let u be a state in G^2_S with weight $(\boldsymbol{x})_u \geq 12$. A 16-full splitting of u results in at least 12 descendant states of weight 1. We claim that at least one of those descendant states, say $u^{(1)}$, inherits only one outgoing edge of u in G^2_S; otherwise, there had to be at least $2 \cdot 12 = 24$ outgoing edges of u in G^2_S, thereby contradicting the upper bound, 21, on the out-degree of u. Now, the terminal state, v, of the edge inherited by $u^{(1)}$ has weight $(\boldsymbol{x})_v \leq 15$. This, however, implies that the 16-full splitting of G^2_S is not $\boldsymbol{x}$-consistent.

Case 2: $\|\boldsymbol{x}\| = 16$. Using Franaszek's algorithm we find that every $(A_{G_S}, 4)$-approximate eigenvector $\boldsymbol{x}$ with $\|\boldsymbol{x}\| = 16$ satisfies

$$\boldsymbol{x} \leq \boldsymbol{\xi} = [\, 9\ 12\ 16\ 12\ 16\ 12\ 16\ 11\ 15\ 11\ 15\ 11\ 14\ 10\ 13\ 8\ 11\ 5\ 7\,]^\top . \tag{6}$$

It follows from Figure 11 that each state in G^2_S has at most six outgoing edges whose terminal states v have weight $(\boldsymbol{x})_v = 16$. Let u be a state in G^2_S of weight $(\boldsymbol{x})_u = 16$. The 16-full splitting of u results in 16 descendant states, each of weight 1. Let s be the number of those descendant states that inherit only one outgoing edge of u; this means that the remaining $16-s$ descendant states inherit at least two edges each. It follows that the number of outgoing edges of u in G^2_S is at least $2 \cdot (16 - s) + s = 32 - s$. This implies $s \geq 11$, since the out-degree of u in G^2_S is at most 21. In particular, there are at least $11 - 6 = 5$ descendant states of u each inheriting a single edge of u with a terminal state, v, that has weight $(\boldsymbol{x})_v < 16$. This again implies that the 16-full splitting of G^2_S is not $\boldsymbol{x}$-consistent. ∎

The proof of Theorem 2.5, as it appears in [4], can be modified and extended to show that the (A_G, n)-approximate eigenvector $\boldsymbol{x}$ in the theorem also satisfies the bound $\|\boldsymbol{x}\| \leq n^{\mathsf{a}}$. When this additional condition is taken into account, then the results in Table 1 can be obtained also from this extension of the theorem. Nevertheless, Example 6.4 below shows that such an extension of Theorem 2.5 is still weaker than Theorem 3.2 (and hence weaker than Theorem 3.1).

Example 6.4 Consider the graph $\mathcal{H}$ in Figure 12, where we assume that the edges have distinct labels. It can be verified that $\lambda(A_{\mathcal{H}}) = 2$ and that $\boldsymbol{\xi} = [\,4\ 2\ 1\ 2\ 3\,]$ is an $(A_{\mathcal{H}}, 2)$-integral eigenvector (the components of $\boldsymbol{x}$ index the names of the states in the figure). Since $\mathcal{H}$ is irreducible, it follows by the Perron-Frobenius Theorem that every $(A_{\mathcal{H}}, 2)$-approximate eigenvector is a scalar multiple of $\boldsymbol{\xi}$ (see [15, Theorem 3.6]).

We show that there is no $(S(\mathcal{H}), 2)$-encoder with anticipation $\mathsf{a} = 2$. Suppose to the contrary that there is such an encoder and apply Theorem 3.2 with $G = \mathcal{H}$ and $k = 1$. From Theorem 3.2(i) it follows that $\boldsymbol{\xi}$ is the only possible choice for the $(A_G, 2)$-approximate eigenvector $\boldsymbol{x}$ in that theorem. However, it is easy to see that there is no $\boldsymbol{\xi}$-consistent splitting of state α in $\mathcal{H}$ into two descendant states of weight 2. Hence the contradiction.

It can be verified, however, that $\mathcal{H}^2$ *can* be 2^2-fully split in one round consistently with $\boldsymbol{\xi}$. This means that we will not reach a contradiction if we attempt to apply Theorem 3.2 with $k = \mathsf{a} = 2$. In other words, Theorem 2.5 does not rule out anticipation 2 in this example. ■

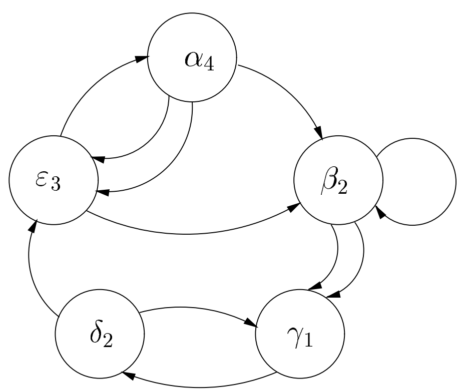

Figure 12: Graph $\mathcal{H}$ for Example 6.4

7 Conclusion

In this work, we presented lower bounds on the anticipation of encoders for input-constrained channels—in the general case and in three particular cases of practical value that are summarized in Table 1. We also demonstrated the universality of the state-splitting algorithm with respect to encoders with finite anticipation: every finite-state encoder with finite anticipation can be obtained by state-splitting operations, followed by a reduction of states.

Our main results are Theorem 3.1 and Theorem 3.2. Theorem 3.1 provides a *necessary and sufficient* condition on the existence of (S, n) encoders with anticipation a, for given S, n, and a. As shown in Section 6.1, the condition may be considered as a constructive criterion for verifying the existence of (S, n) encoders with anticipation a, though the verification procedure may be impractical. Finding an efficient procedure that constructs encoders with *minimal* anticipation through the state-splitting algorithm requires further research.

Unlike Theorem 3.1, Theorem 3.2 gives only *necessary* conditions on the existence of (S, n)-encoders with anticipation a. It is still an open problem whether those conditions are also sufficient. However, since Theorem 3.2 holds for *every* irreducible deterministic presentation G of S, it lends itself more easily than Theorem 3.1 to obtain lower bounds on the anticipation.

A Appendix

Example A.1 demonstrates that sometimes the Shannon cover cannot be taken as the graph $\mathcal{G}$ in Theorem 3.1.

Example A.1 Consider the constraint S presented by the graph H with $V(H) = \{a, b, \ldots, k\}$ and with edges as shown in Figure 13. We assume that all the edges of H are distinctly labeled, so H is deterministic and reduced. It is easy to verify that H is also irreducible; hence, H is the Shannon cover of S.

One can verify that $\lambda(A_H) = 2$ and that the weights that appear as subscripts in Figure 13 form an $(A_H, 2)$-integral eigenvector $\boldsymbol{\xi}$. Any other $(A_H, 2)$-approximate eigenvector is a scalar multiple of $\boldsymbol{\xi}$ (see [15, Theorem 3.6]). Since $\|\boldsymbol{\xi}\| = 2^4 = 16$ it follows from Theorem 2.4 (or from Theorem 3.1(i) combined with Lemma 6.2) that the anticipation of every $(S, 2)$-encoder is at least 4. In what follows we show that $(S, 2)$-encoders with anticipation $\mathsf{a} = 4$ do exist; yet, those encoders cannot be obtained by substituting $\mathcal{G} = H$ in Theorem 3.1.

Assume to the contrary that $\mathcal{G} = H$ which, by Theorem 3.1(i), implies that the $(A_{\mathcal{G}}, 2)$-approximate eigenvector $\boldsymbol{x}$ must be $\boldsymbol{\xi}$. By Theorem 3.1(iii), each state in $\mathcal{G}$ is split in the first round into one or two descendant states, and by Theorem 3.1(iv), the induced approximate eigenvector, $\boldsymbol{x}^{(1)}$, satisfies $\|\boldsymbol{x}^{(1)}\| \le 2^3 = 8$. States a, b, and c are split in the first round into two descendant states, since each of the weights $(\boldsymbol{\xi})_a$, $(\boldsymbol{\xi})_b$, and $(\boldsymbol{\xi})_c$ is greater than 8.

State a has weight $(\boldsymbol{\xi})_a = 16$, so it is split into two descendants states, $a^{(1)}$ and $a^{(2)}$, with weights $(\boldsymbol{x}^{(1)})_{a^{(1)}} = (\boldsymbol{x}^{(1)})_{a^{(2)}} = 8$. Figure 14(a) shows the only possible $\boldsymbol{\xi}$-consistent partition of the outgoing edges of a in H. Similarly, state b is split into two descendants states, $b^{(1)}$ and $b^{(2)}$, each of weight 8, and the only possible $\boldsymbol{\xi}$-consistent partition of the outgoing edges of b in H in shown in Figure 14(b). As for state c, there are two possible ways to partition its outgoing edges consistently with $\boldsymbol{\xi}$; those two possible partitions are shown in Figure 15(a) and 15(b). Note that each of the states d and e has only one outgoing edge in H; hence, those states cannot be split in the first round.

Figure 16(a) (respectively, Figure 16(b)) shows the outgoing edges of $a^{(2)}$ and $b^{(2)}$ after the first round, given that c was split according to the partition in Figure 15(a) (respectively, Figure 15(b)). Now, by Theorem 3.2(iii)–(iv), each of the states $a^{(2)}$ and $b^{(2)}$ must be split

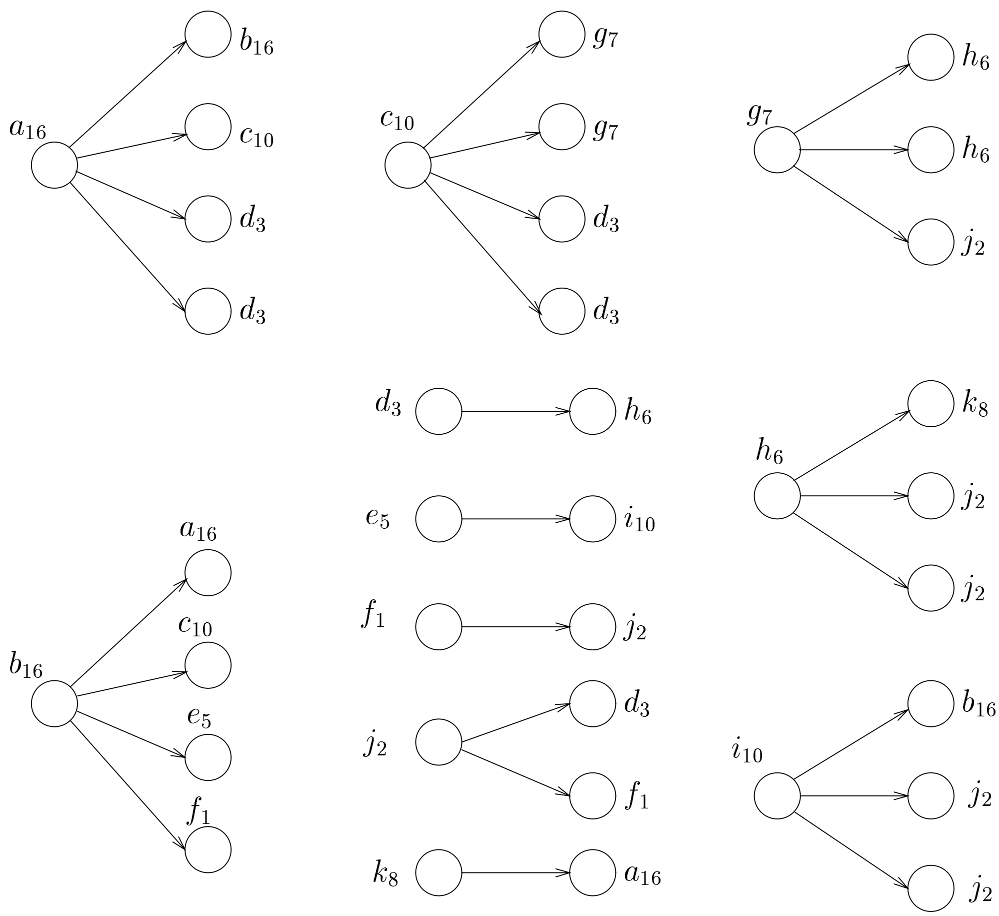

Figure 13: Edges of graph H for Example A.1.

in the second round into two descendant states of weight 4. Yet, given those weights of the descendant states, the outgoing edges of state $a^{(2)}$ in Figure 16(a) cannot be partitioned into two sets consistently with $\boldsymbol{x}^{(1)}$, neither can the outgoing edges of state $b^{(2)}$ in Figure 16(b). Hence the contradiction. We therefore conclude that the graph $\mathcal{G}$ in Theorem 3.1 cannot be the Shannon cover H.

It turns out that by duplicating state c, we can obtain an expansion $\tilde{H}$ of H that may serve as the graph $\mathcal{G}$ in Theorem 3.1 for $\mathsf{a} = 4$. The new duplicate state, $\tilde{c}$, has weight 10, and the edge $b \to c$ in H is redirected in $\tilde{H}$ into $\tilde{c}$. Indeed, in the first round of splitting, state c is split into $c^{(1)}$ and $c^{(2)}$ with weights $(\boldsymbol{x}^{(1)})_{c^{(1)}} = (\boldsymbol{x}^{(1)})_{c^{(2)}} = 5$, and $\tilde{c}$ is split into $\tilde{c}^{(1)}$ and $\tilde{c}^{(2)}$ with weights $(\boldsymbol{x}^{(1)})_{\tilde{c}^{(1)}} = 3$ and $(\boldsymbol{x}^{(1)})_{\tilde{c}^{(2)}} = 7$. A rather straightforward check reveals that we can now continue with three more rounds to obtain a 2-full splitting of $\tilde{H}$, where in the ith round the induced approximate eigenvector $\boldsymbol{x}^{(i)}$ satisfies $\|\boldsymbol{x}^{(i)}\| = 2^{4-i}$, and each state is split into one or two descendant states. ■

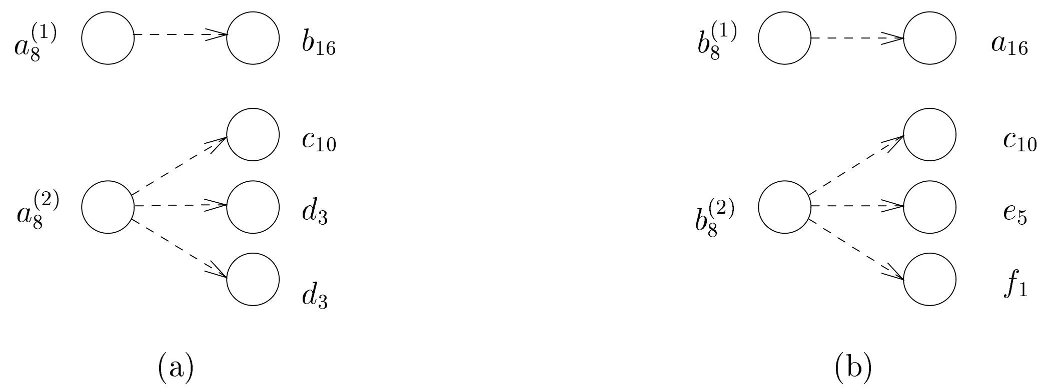

Figure 14: Splitting of states a and b in the first round.

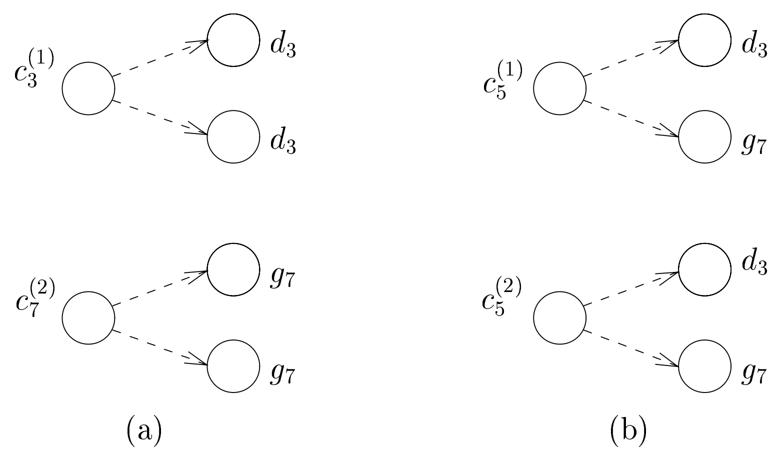

Figure 15: Possible partitions of outgoing edges of state c.

B Appendix

Example B.1 An additional proof is given below for the result in Example 6.3. It is shown that no deterministic presentation $\mathcal{G}$ of $S^5_{2,18,2}$ can turn into an $(S^5_{2,18,2}, 2^2)$-encoder with anticipation 2 by *two rounds* of splitting that satisfy the requirements of Theorem 3.1.

Assume to the contrary that there is an $(S^5_{2,18,2}, 2^2)$-encoder with anticipation $\mathsf{a} = 2$ and let $\mathcal{G}$ and $\boldsymbol{x}$ be as in Theorem 3.1 (with $n = 2^2 = 4$). In particular, by Theorem 3.1(i) we have $\|\boldsymbol{x}\| \leq 16$. Franaszek's algorithm and Lemma 6.2 imply that $\|\boldsymbol{x}\| \geq 12$. We denote by $\boldsymbol{x}^{(1)}$ the induced approximate eigenvector after the first $\boldsymbol{x}$-consistent round of splitting of $\mathcal{G}$; recall that from Theorem 3.1(iv) we have $\|\boldsymbol{x}^{(1)}\| \leq n = 4$. We distinguish between the following two cases:

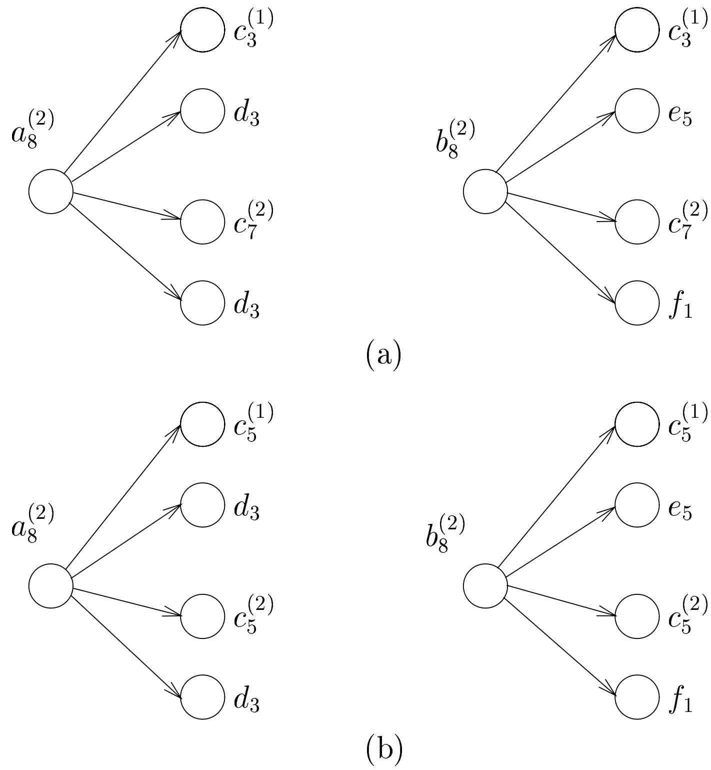

Figure 16: Possible configurations of outgoing edges of states $a^{(2)}$ and $b^{(2)}$ after the first round.

Case 1: $12 \le \|\boldsymbol{x}\| \le 15$. Let u be a state in $\mathcal{G}$ with weight $(\boldsymbol{x})_u \ge 12$ and let $\{u^{(r)}\}_{r=1}^{N(u)}$ be the set of descendant states of u after the first round of splitting. By Theorem 3.1(iii) we have $N(u) \le 4$. On the other hand,

$$4 \cdot N(u) \ge \|\boldsymbol{x}^{(1)}\| \cdot N(u) \ge \sum_{r=1}^{N(u)} (\boldsymbol{x}^{(1)})_{u^{(r)}} = (\boldsymbol{x})_u \ge 12 \ ,$$

thereby implying that $N(u) \ge 3$. We first rule out the case $N(u) = 3$. Indeed, if $N(u) = 3$ we would have $(\boldsymbol{x}^{(1)})_{u^{(r)}} = 4$ for $r \in \{1, 2, 3\}$, and at least one descendant state, say $u^{(1)}$, would inherit only one of the (at most five) outgoing edges of u in $\mathcal{G}$, say edge e. This, in turn, would imply the contradiction

$$15 \ge \|\boldsymbol{x}\| \ge (\boldsymbol{x})_{\tau_{\mathcal{G}}(e)} \ge n \cdot (\boldsymbol{x}^{(1)})_{u^{(1)}} = 16 \ .$$

Hence, $N(u) = 4$.

As the out-degree of u in $\mathcal{G}$ is at most 5, at least three of the four descendant states of u, say $u^{(1)}, u^{(2)}, u^{(3)}$, inherit a single edge each. For those states we have $n \cdot (\boldsymbol{x}^{(1)})_{u^{(r)}} \le \|\boldsymbol{x}\| \le 15$ and, so, $(\boldsymbol{x}^{(1)})_{u^{(r)}} \le 3$. On the other hand, since $\sum_{r=1}^{4} (\boldsymbol{x}^{(1)})_{u^{(r)}} = (\boldsymbol{x})_u \ge 12$

and $(\boldsymbol{x}^{(1)})_{u^{(4)}} \leq 4$, it follows that $(\boldsymbol{x}^{(1)})_{u^{(r)}} = 3$ for at least two of the descendant states $u^{(1)}, u^{(2)}, u^{(3)}$. Without loss of generality we assume that $(\boldsymbol{x}^{(1)})_{u^{(1)}} = 3$.

Let $u \to v$ be the (only) outgoing edge of u in $\mathcal{G}$ that is inherited by $u^{(1)}$. From

$$(\boldsymbol{x})_v \geq n \cdot (\boldsymbol{x}^{(1)})_{u^{(1)}} = 4 \cdot 3 = 12$$

we can apply the foregoing discussion on u also to v; in particular, v must also be split in the first round into $N(v) = 4$ descendant states $\{v^{(r)}\}_{r=1}^{N(v)}$, with weights $(\boldsymbol{x}^{(1)})_{v^{(1)}} = (\boldsymbol{x}^{(1)})_{v^{(2)}} = 3$ and $(\boldsymbol{x}^{(1)})_{v^{(3)}} \leq 3$. The three possible configurations of weights of terminal states of the outgoing edges of state $u^{(1)}$ are shown in Figure 17(a)–(c).

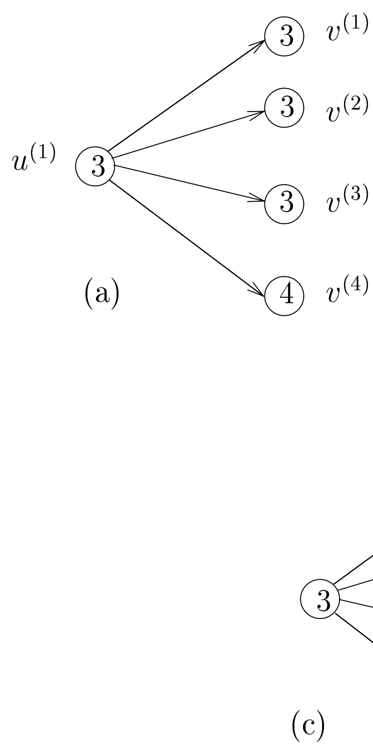

Figure 17: Possible configurations of outgoing edges of state $u^{(1)}$ after the first round.

In the second (and last) round, $u^{(1)}$ is 4-fully split into three descendant states. As $u^{(1)}$ has out-degree 4, two of its descendant states inherit a single edge each; furthermore, one of those single edges is $u^{(1)} \to v^{(r)}$ where $(\boldsymbol{x}^{(1)})_{v^{(r)}} \leq 3$. This, however, makes the second round inconsistent with $\boldsymbol{x}^{(1)}$.

Case 2: $\|\boldsymbol{x}\| = 16$. By Franaszek's algorithm, every $(A_{G_S}, 4)$-approximate eigenvector $\boldsymbol{x}'$ with $\|\boldsymbol{x}'\| = 16$ satisfies $\boldsymbol{x}' \leq \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is defined in (6). Now, from Figure 11 we see that every state in G_S has at most two outgoing edges that terminate in states v_i with weights $(\boldsymbol{\xi})_{v_i} = 16$. This property will be preserved if $\boldsymbol{\xi}$ is replaced by an $(A_{G_S}, 4)$-approximate eigenvector $\boldsymbol{x}'$ with $\|\boldsymbol{x}'\| = 16$. It follows from Lemma 6.2 that every state in $\mathcal{G}$ has at most two outgoing edges that terminate in states v with weights $(\boldsymbol{x})_v = 16$.

Let u be a state in $\mathcal{G}$ with weight $(\boldsymbol{x})_u = 16$ and let $\{u^{(r)}\}_{r=1}^{N(u)}$ be the set of descendant states of u after the first round of splitting. By Theorem 3.1(iii)–(iv) we have $N(u) = 4$. and $(\boldsymbol{x}^{(1)})_{u^{(r)}} = 4$ for $r = 1, 2, 3, 4$. As the out-degree of u in $\mathcal{G}$ is at most 5, at least three of these four descendant states inherit a single edge each. Furthermore, since there are at most two outgoing edges of u that terminate in states v with weight $(\boldsymbol{x})_v = 16$, it follows that at least one of the descendant states of u, say $u^{(1)}$, gets a single edge $u \to v$ where $(\boldsymbol{x})_v < 16$. This, however, contradicts the inequality $(\boldsymbol{x})_v \geq n \cdot (\boldsymbol{x}^{(1)})_{u^{(1)}} = 16$. ■

References

[1] R.L. Adler, D. Coppersmith, M. Hassner, *Algorithms for sliding block codes—an application of symbolic dynamics to information theory, IEEE Trans. Inform. Theory,* 29 (1983), 5–22.

[2] J.J. Ashley, B.H. Marcus, *Canonical encoders for sliding block decoders, SIAM J. Discrete Math.,* 8 (1995), 555–605.

[3] J.J. Ashley, B.H. Marcus, R.M. Roth, *Construction of encoders with small decoding look-ahead for input-constrained channels, IEEE Trans. Inform. Theory,* 41 (1995), 55–76.

[4] J.J. Ashley, B.H. Marcus, R.M. Roth, *On the decoding delay of encoders for input-constrained channels, IEEE Trans. Inform. Theory*, 42 (1996), 1948–1956.

[5] J.J. Ashley, *Performance Bounds in Constrained Sequence Coding,* Ph.D. Thesis, University of California, Santa Cruz, 1987.

[6] J.J. Ashley, *A linear bound for sliding block decoder window size, IEEE Trans. Inform. Theory,* 34 (1988), 389–399.

[7] J.J. Ashley, *A linear bound for sliding block decoder window size, II, IEEE Trans. Inform. Theory,* 42 (1996), 1913–1924.

[8] P.A. Franaszek, *Run-length-limited variable length coding with error propagation limitation,* US patent 3,689,899 (1972).

[9] H.D.L. Hollmann, *On the construction of bounded-delay encodable codes for constrained systems, IEEE Trans. Inform. Theory,* 41 (1995), 1354–1378.

[10] T.D. Howell, *Statistical properties of selected recording codes, IBM J. Res. Dev.,* 32 (1989), 60–73.

[11] K.A.S. Immink, *Coding Techniques for Digital Recorders,* Prentice Hall, New York, 1991.

[12] K.A.S. Immink, *EFMPlus: The coding format of the multimedia compact disc, IEEE Trans. Consum. Electron.,* 41 (1995), 491–497.

[13] H. Kamabe, *Minimum scope for sliding block decoder mappings, IEEE Trans. Inform. Theory,* 35 (1989), 1335–1340.

[14] B.H. Marcus, R.M. Roth, *Bounds on the number of states in encoder graphs for input-constrained channels, IEEE Trans. Inform. Theory,* 37 (1991), 742–758.

[15] B.H. Marcus, R.M. Roth, P.H. Siegel, *Constrained systems and coding for recording channels, Handbook of Coding Theory,* V.S. Pless and W.C. Huffman (Editors), Elsevier, Amsterdam, 1999, pp. 1635–1764.

[16] G. Ruckenstein (Sadeh), *Encoding for Input-Constrained Channels,* M.Sc. Thesis, Computer Science Department, Technion, Haifa, Israel, 1996.

[17] D. Rugar, P.H. Siegel, *Recording results and coding considerations for the resonant bias coil overwrite technique, Optical Data Storage Topical Meeting,* G.R. Knight, C.N. Kurtz (Editors), *Proc. SPIE,* 1078 (1989), 265–270.

[18] C.E. Shannon, *The mathematical theory of communication, Bell Sys. Tech. J.,* 27 (1948), 379–423.

[19] T.C. Weigandt, *Magneto-optic Recording Using a* $(2,18,2)$*-Run-Length-Limited Code,* Master Thesis, Massachusetts Institute of Technology, 1990.

[20] A.D. Weathers, J.K. Wolf, *A new 2/3 sliding block code for the* $(1,7)$ *runlength constraint with the minimal number of encoder states, IEEE Trans. Inform. Theory*, 37 (1991), 908–913.